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ARTICLE

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HAMILTONIAN STRUCTURE OF THE CYLINDRICAL KORTEWEG-DE VRIES EQUATION FOR DUST ION ACOUSTIC WAVES IN UNMAGNETISED PLASMA

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By employing reduction perturbation technique (RPT), the cylindrical Korteweg-de Vries equation (cKdV) for dust ion acoustic waves in unmagnetised plasma is derived. Using Hamilton's variational principle, the expressions for Lagrangian densities of the cKdV fields are constructed. The expressions for Lagrangian densities can be used to construct the Hamiltonian densities which characterize the Zakaraov-Faddeev-Gardner equation.

Introduction

In the recent past, the dust ion acoustic solitary waves (DIASWs) have received substantial attentions by several authors[1-3]. One applies RPT to study such waves in the frame work of the KdV and deformed KdV (dKdV) equations[4-8]. It is important to note that most of the studies are confined to one-dimensional (planar) geometry. DIASWs in bounded (spherical and cylindrical) plasmas significantly differ from those in unbounded (planar) one. Recent studies show that the one-dimensional geometry cannot account for all features of laboratory plasma. The results so obtained from laboratory experiments do not match with that of the planar one. Therefore, it is worthwhile to study DIASWs in bounded geometry. In practice, all laboratory instruments being spherical or cylindrical in shape, so one needs an adequate theory that contains this spherical or cylindrical correction $\left[\frac{\gamma}{2\eta}\varphi^{(1)}\right]$ in the KdV equation. So far, several attempts have been made to work out a solution of spherical or cylindrical KdV equation through an ad-hoc approach. These are purely numerical improvements of the solutions of the KdV equation in plasma geometry. But real situation demands the solution of nonlinear evolution equation in bounded (non-planar) geometry. This problem has been addressed in a systematic way and analytical solution is obtained by Lie symmetry approach[9]. Here I have derived the deformed KdV equation and have got an analytical solution which was not attempted before.

Maxon and Viecelli[10] reported the first work on solitons having dimension greater than one. Mamun and Shukla[11] studied the DIASWs in non-planar cylindrical and spherical geometry. But they used the stationary solution of the planar equation as the initial condition and improved the result numerically and obtained pure soliton-like solution without the non-planar effect in both the cases[10,11]. Sahu and Roychoudhury[12-13] also analyzed solitary waves numerically in non-planar geometry in different plasma models. In the recent past, Sahu and Roychoudhury[14] made a venture to obtain an analytical solution of the cylindrical and spherical KdV equations by Lie algebraic technique without non-planar effect. However, the results so obtained do not exhibit the non-planar effect. The invariance of the nonlinear PDEs with respect to one parameter Lie group of point transformations plays a substantial role to obtain exact analytical solutions of the cKdV equation[9].

The objective of the present paper is two-fold. First, the cKdV equation for the DIASWs in cylindrical geometry

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by the method of RPT is derived. Secondly, the cKdV equation is transformed by the invertible point transformation to the autonomous form[15] and the inverse problem of variational calculus for cKdV equation is solved to obtain the expressions for Hamiltonian densities that characterize the Zakharov-Faddeev-Gardner equation.

Derivation of Cylindrical Kdv Equations:

The DIASWs in a non-planar cylindrical geometry are given by

$$\partial_t n_i + \frac{1}{r^\gamma}\partial_r\left(r^\gamma n_i u_i\right) = 0 \quad (1)$$

$$\partial_t u_i + u_i \partial_r u_i - \partial_r \varphi = 0 \quad (2)$$

$$\frac{1}{r^\gamma}\partial_r\left(r^\gamma \partial_r \varphi\right) = \mu \exp(\varphi) - n_i + (1-\mu) \quad (3)$$

The continuity, momentum and Poisson's equations are represented by equations (1-3) respectively. Here, $\partial_t = \frac{\partial}{\partial t}$ and $\partial_r = \frac{\partial}{\partial r}$. The ion-fluid speed u_i is normalized by $c_i = \left(\frac{k_B T_e}{m_i}\right)^{\frac{1}{2}}$ where k_B is the Boltzmann constant and T_e is the electron temperature and m_i is the ion mass respectively. The space variable r and time variable t are normalized by $\lambda_D = \left(\frac{k_B T_e}{4\pi n_{i0} e^2}\right)^{\frac{1}{2}}$ and the ion-plasma period $\omega_{pi}^{-1} = \left(\frac{m_i}{4\pi n_{i0} e^2}\right)^{\frac{1}{2}}$ respectively. n_i, is the ion density normalized by its equilibrium value n_{i0}. $\mu = \frac{n_{e0}}{n_{i0}}$, where, n_{e0} and n_{i0} are the particle number density for electrons and ions respectively. φ being the electrostatic wave potential normalized by $\frac{k_B T_e}{e}$. Here, γ = 0, stands for one-dimensional geometry and γ = 1, 2 for cylindrical or spherical geometry (non-planar) respectively.

In order to solve eqns. (1-3), in the small amplitude approximation, one introduces the stretched coordinates

$$\xi = -\xi^{\frac{1}{2}}(r + v_0 t) \text{ and } \eta = \xi^{\frac{3}{2}} t \quad (4)$$

According toRPT, the dependent variables are expanded as

$$n_i = 1 + \varepsilon n_i^1 + \varepsilon^2 n_i^2 + \varepsilon^3 n_i^3 + \ldots\ldots., \quad (5)$$

$$u_i = \varepsilon u_i^1 + \varepsilon^2 u_i^2 + \varepsilon^3 u_i^3 + \ldots\ldots., \quad (6)$$

$$\varphi = \varepsilon \varphi^1 + \varepsilon^2 \varphi^2 + \varepsilon^3 \varphi^3 + \ldots\ldots, \quad (7)$$

The time and space derivatives are transformed to

$$\partial_t = -\varepsilon^{\frac{1}{2}}\partial_\xi + \varepsilon^{\frac{3}{2}}\partial_\eta \quad (8)$$

$$\partial_r = -\varepsilon^{\frac{1}{2}}\partial_\xi \quad (9)$$

From the transformation, we have

$$r = -\varepsilon^{-\frac{3}{2}}\left(\varepsilon\xi + \eta\right) \quad (10)$$

Using eqns. (5-10) in eqns. (1-3) and equating the coefficients of the lowest power of ε one can have

$$v_0 = \frac{1}{\sqrt{\mu}} \quad (11)$$

It is worthwhile to note that the reductive perturbation method is generally misleading because of its over-restrictive scaling. However, one obtains the correct deformed KdV equation by taking recourse of the use of this method as given by

$$\partial_\eta \varphi^1 + \frac{\gamma}{2\eta}\varphi^1 + a\varphi^1 \partial_\xi \varphi^1 + b\partial_{3\xi}\varphi^1 = 0\,, \quad (12)$$

here,

$$a = \frac{\sqrt{\mu}}{2}\left(3 - \frac{1}{\mu}\right) \quad (13)$$

$$b = \frac{1}{2}\mu^{-\frac{3}{2}}$$

As mentioned before that for one dimensional (planar) geometry $\gamma = 0$ and for a non-planar geometry (cylindrical or spherical) γ = 1 or 2 respectively. It is believed that an exact analytical solution of eqn. (12) is hardly possible and it is an obvious consequence to have numerical solution[10]. A novel attempt has been made to have an exact analytical solution of the eqn. (12) by Ghosh et al[9].

The following transformations in eqn. (14) reduce the dKdV eqn. (12) to the standard cKdV equation eqn. (15) which is supposed to describe cylindrical solitons.

$$x = \xi b^{-\frac{1}{3}};\; u = ab^{-\frac{1}{3}}\varphi^1 \text{ and } t = \eta \quad (14)$$

$$u_t + uu_x + u_{3x} + \frac{u}{2t} = 0 . \quad (15)$$

It is obvious to note that without the term $\frac{u}{2t}$ eqn. (15) in question is nothing but the KdV equation. Eqn. (15) is a good approximation to the system of eqns. (1-5) for weak dispersion and weak nonlinearity.

Exact Solution of the cKdV Equation for Dust Ion Acoustic Wave:

The infinitesimal generators or transform operators of the cKdV eqn. (15) are given by[16]

$$X_1 = \partial_x \quad (16)$$

$$X_2 = x\partial_x + 3t\partial_t - 2u\partial_u ; \quad (17)$$

$$X_3 = 2\sqrt{t}\partial_x + \frac{1}{\sqrt{t}}\partial_u ; \quad (18)$$

$$X_4 = 2x\sqrt{t}\partial_x + 4t\sqrt{t}\partial_t + \left(\frac{x}{\sqrt{t}} - 4u\sqrt{t}\right)\partial_u \quad (19)$$

The idea of the Lie point symmetries admitted by a system of PDEs may be used to obtain invariant solutions of the non-planar (cylindrical or spherical) system. $\mathcal{F}(x, t)$ is said to be an invariant of the equations if and only if $X_i\mathcal{F} = 0$, ($i = 1,2,3,4$) is satisfied. It is shown that the eqn. (15) is left invariant by the Lie groups of point transformations generated by the two commuting generators as given in eqns. (18) and (19). The commutators of Lie algebra are given in table 1 whose (i, j)th element is $\left[X_i, X_j\right]$. It is obvious[17] that the commutator of any two infinitesimal generators of an r – parameter Lie group of transformations is also an infinitesimal generator. In particular,

$$\left[X_i, X_j\right] = \sum_{k=1}^{r} C_{ij}^{k} X_k ; \quad i, j = 1,2....r. \quad (20)$$

where the coefficients are constants known as structure constants

Table1

	X_1	X_2	X_3	X_4
X_1	0	X_1	0	X_3
X_2	$-X_1$	0	$\frac{1}{2}X_3$	$\frac{3}{2}X_4$
X_3	0	$-\frac{1}{2}X_3$	0	0
X_4	$-X_3$	$-\frac{3}{2}X_4$	0	0

It is obvious that the table is anti-symmetric with all its diagonal elements zero. One can easily read off the structure constants from the commutator table. It is obvious[18] that the cKdV system can be reduced to the standard KdV equation.

Let me introduce the canonical variables τ, ξ, q in terms of which the infinitesimal generators X_3 and X_4 are defined by

$$\left.\begin{array}{l} X_3\xi = 1, \\ X_3\tau = 0, \\ X_3 q = 0. \end{array}\right\} \text{ and } \left.\begin{array}{l} X_4\tau = 1, \\ X_4\xi = 0, \\ X_4 q = 0. \end{array}\right\} \quad (21)$$

The infinitesimal generators (X_3, X_4) are easily reduced to

$$\begin{array}{l} X_3 = \partial_\xi \\ X_4 = \partial_\tau \end{array} \quad (22)$$

It is an obvious interest to note that it merely corresponds to a translation in the variables (τ, ξ). It can also be noticed from the commutator table that the infinitesimal generators X_3 and X_4 admitted by the system (15), commute. This implies that the generators X_3 and X_4 generate a 2-dimemsional abelian sub-algebra and hence the system (15) can be transformed by the invertible point transformation:

$$\tau = -2t^{-\frac{1}{2}}, \ \xi = \frac{1}{2}t^{-\frac{1}{2}} \text{ and } q = ut - \frac{x}{2} \quad (23)$$

to the following autonomous form[15]

$$q_\tau + qq_\xi + q_{3\xi} = 0 . \quad (24)$$

This is the standard KdV equation in the variables (τ, ξ) with the following exact solution

$$q = 12 \operatorname{sech}^2 \left[\xi - 4\tau\right] \quad (25)$$

The invertible transformations (23) in conjunction with eqn. (25) lead to the following solution of the system (15)

$$u = \frac{x}{2t} + \frac{12}{t}\operatorname{sech}^2\left[\frac{x+8}{\sqrt{t}}\right] \quad (26)$$

It is obvious that eqn. (26) is a nontrivial solution of the original system (15). Thus, in view of the equations (15) and (26), an exact solution of the cKdV eqn. (12) can be expressed as:

$$\varphi^1 = \frac{\xi}{2a\eta} + \frac{12b^{\frac{1}{3}}}{a\eta}\operatorname{sech}^2\left[\frac{\xi + 8b^{\frac{1}{3}}}{\sqrt{\eta}\, b^{\frac{1}{3}}}\right] \tag{27}$$

It is worth mentioning that the solution (27) of the deformed KdV equation (12) is not a pure soliton solution as it contains a non-soliton part $\left(= \frac{\xi}{2a\eta}\right)$ This is because of the term $\frac{\gamma}{2\eta}\varphi^{(1)}$ of the deformed KdV equation which is due to the effect of the non-planar (cylindrical or spherical) geometry. Figure 1 presents the evolution of $\varphi^{(1)}$ with η from eqn. (27).

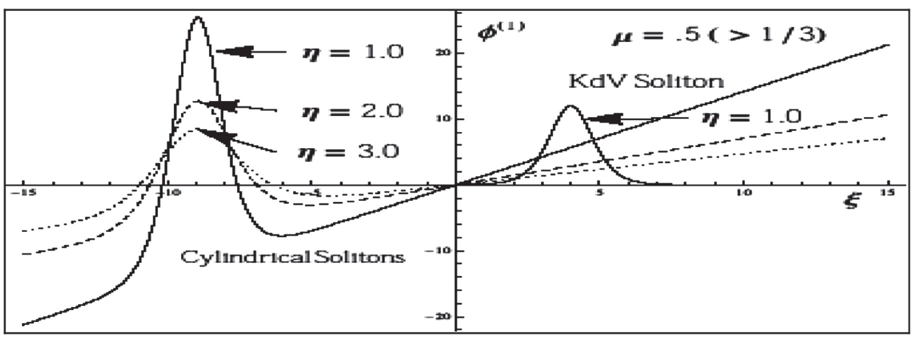

Fig. 1: Depicts the deformed KdV (cylindrical) soliton $\varphi^{(1)}$ versus spatial coordinate ξ at times $\eta = 1.0$, $\eta = 2.0$ and $\eta = 3.0$ for $(\mu = .5)$ and $(\mu = .3)$ respectively.

The analytical solution of (12) implies that for large value of η the cylindrical solitons are similar to one-dimensional deformed KdV soliton. This is because for a large value of η, the term $\frac{\gamma}{2\eta}\varphi^{(1)}$ which is due to the effect of the non-planar (cylindrical or spherical) geometry is no longer important. However, as η increases, the deformed KdV soliton differ from that of the planar one. From the figure 1 also observe that as η decreases, the amplitude of these localized pulses increases. I also notice that the amplitude of the cylindrical soliton is larger than that of the corresponding planar one.

Solution of the Inverse Problem of the cKdV Equation

It is important to note that the deformed KdV equation (12) takes original form of the KdV equation (24) by employing the invertible point transformation. The objective of the present paper is to construct the Hamiltonian structure of cylindrical KdV equation which satisfies the Zakaraov-Faddeev-Gardner equation. Let us start with the equation (24). The factor 6 is just a scaling factor to make solutions easier to describe.

$$q_\tau + q_{3\xi} + 6qq_\xi = 0 \tag{28}$$

Five decades have passed when the KdV equationwas interpreted by Zakharov and Faddeev[19] and by Gardner[20] as a completely integrable Hamiltonian system in an infinite dimensional phase space. The Hamiltonian form of (28) is given by

$$q_\tau = \partial_\xi \frac{\delta \mathcal{H}}{\delta q} \tag{29}$$

with $\partial_\xi = \frac{\partial}{\partial \xi}$ the Hamiltonian operator and $\mathcal{H} = \frac{1}{2}q_\xi^2 - q^3$, the Hamiltonian density. Here $q = q(\xi, \tau)$ and subscripts of q refer to the corresponding partial derivatives. The variational or Euler derivative is given by

$$\frac{\delta}{\delta q} = \frac{\partial}{\partial q} - \frac{\partial}{\partial \xi}\frac{\partial}{\partial q_\xi} + \frac{\partial^2}{\partial \xi^2}\frac{\partial}{\partial q_{2\xi}} + \ldots \tag{30}$$

I approach the problem from a different point of view by solving the inverse problem of variational calculus to get an analytical representation of the system in the coordinate space[21]. All problems of variational calculus fall into one of the two groups. (i) The direct problem, where one first assigns a Lagrangian to the dynamical system and then computes the equations of motion via Euler-Lagrange equation and, (ii) the inverse problem posed in the opposite direction. Here one begins with the equations of motion and tries to derive the Lagrangian for the system. The representation of a system in terms of Euler-Lagrange equations goes by the name analytic representation[21]. Helmholtz theorem[17] serves as a useful tool to solve the inverse problem and thereby study their analytic representation for nonlinear evolution equations. This theorem being derivable from the theory of Lie groups it is both algebraic and geometric in nature. For example, the self-adjointness of the Frechet derivative

$$D_p(Q) = \frac{d}{d\varepsilon} P\left[q + \varepsilon Q[q]\}\right]_{\varepsilon=0} \tag{31}$$

promises the existence of the Lagrangian while the explicit expression for the velocity independent part of the Lagrangian density $\mathcal{L}_2$, is constructed by using the homotopy formula

$$L_2[q] = \int_0^1 qP[\lambda q] d\lambda \tag{32}$$

Here P stands for the Euler-Lagrange expression of the variational problem. I shall make use of eqns. (31&32) in the following to deal with the KdV equation.

One common trick to put a single evolution equation into the variational form is to replace q by a potential function $\omega(\xi, \tau)$ defined by

$$\omega(\xi, \tau) = \int_{\xi}^{\infty} q(y, \tau)\, dy \tag{33}$$

Such that

$$\omega_{\xi}(\xi, \tau) = -q(\xi, \tau) \tag{34}$$

It is interesting to note that this trick works only for equations which are of odd order in space derivatives and, fortunately for us, KdV equation belongs to this class. Employing eqn. (34) in (28), I get

$$\omega_{\xi\tau} + \omega_{4\xi} + 6\omega_{\xi}\omega_{2\xi} = 0 \tag{35}$$

From (35), the Euler-Lagrange expression

$$P(\omega) = \omega_{4\xi} + 6\omega_{\xi}\omega_{2\xi} \tag{36}$$

using eqn. (31), gives the self-adjoint operator as given below.

$$\mathcal{D}_P = \mathcal{D}_{\xi}^4 + 6\omega_{2\xi}\mathcal{D}_{\xi} + 6\omega_{\xi}\mathcal{D}_{\xi} \tag{37}$$

The self-adjointness of eqn. (37) confirms the existence of the Lagrangian for eqn. (28). The Lagrangian density $\mathcal{L}_2$ for $P(\omega)$ in (36) can be calculatedfrom (32) to give

$$\mathcal{L}_2 = \frac{1}{2}\omega\omega_{4\xi} + 2\omega\omega_{\xi}\omega_{2\xi} \tag{38}$$

A velocity (ω_{τ}) dependent part, $\mathcal{L}_1 = \frac{1}{2}\omega_{\tau}\omega_{\xi}$ is added to $\mathcal{L}_2$ to get the total Lagrangian density for eqn. (35) and I have

$$\mathcal{L}_h = \frac{1}{2}\omega_{\tau}\omega_{\xi} + \frac{1}{2}\omega\omega_{4\xi} + 2\omega\omega_{\xi}\omega_{2\xi} \tag{39}$$

The expression in (39) is of first order in time and fourth order in space. It is worthwhile to note that the Lagrangian density for (28) or (35) can also be obtained without making use of the homotopy formula (32). To see this I integrate(35) with respect to ξ and write

$$\omega_{\tau} + \omega_{3\xi} + 3\omega_{\xi}^2 = 0 \tag{40}$$

Here I have made use of the boundary conditions $q(\xi, \tau) \to 0$ as $\xi \to \pm\infty$. It is easy to express (40) in the variationalform[22]

$$\delta\int_{\tau_1}^{\tau_2} d\tau \int_{-\infty}^{\infty} d\xi\, \mathcal{L}\left(\omega_{\tau}\omega_{\xi}\omega_{2\xi}\right) = 0 \tag{41}$$

with the Lagrangian density

$$\mathcal{L}_v = \frac{1}{2}\omega_{\tau}\omega_{\xi} - \frac{1}{2}\omega_{2\xi}^2 + \omega_{\xi}^3 \tag{42}$$

Here the subscripts h and v of $\mathcal{L}$ refer to Lagrangian densities obtained from the homotopy formula and direct use of variational principle. As opposed to the result for $\mathcal{L}_h$, the expression for $\mathcal{L}_v$ is only of second order in space derivative. However, $\mathcal{L}_h$ and $\mathcal{L}_v$ are equivalent as they differby a gauge function and one can write

$$\mathcal{L}_h = \mathcal{L}_v + \partial_{\xi}\left(\frac{1}{2}\omega_{\tau}\omega_{3\xi} - \frac{1}{2}\omega_{\xi}\omega_{2\xi} - \omega\omega_{\xi}^2\right) \tag{43}$$

Both the Lagrangians, when substitutedin the appropriate Euler-Lagrange equations reproduce the evolution equations in (40).

Both the Lagrangians lead to the canonical momentum density

$$\pi = \frac{1}{2}\omega_{\xi} \tag{44}$$

This equation cannot be inverted for the velocity ω_{τ} implying that the Lagrangian densities are degenerate[23]. Therefore, one must use the Dirac's theory of constraints[24] to obtain the total Hamiltonian density given by

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \tag{45}$$

Here $\mathcal{H}_0$ is the free part of $\mathcal{H}$ determined by the usual Legendre map and evaluation of the expression for $\mathcal{H}_1$ requires the explicit use of Dirac's theory of constraints. It is of obvious interestto note that the free part $\mathcal{H}_0$ only in conjunction with equations (29, 30, 33) reproduces the KdV equation in (40) written[25] in terms of ω. The Hamiltonian densities $\mathcal{H}_{0h}$ and $\mathcal{H}_{0v}$ corresponding to $\mathcal{L}_h$ and $\mathcal{L}_v$ can be obtained as

$$\mathcal{H}_{0h} = -\frac{1}{2}\omega\omega_{4\xi} - 2\omega\omega_{\xi}\omega_{2\xi} \tag{46}$$

and

$$\mathcal{H}_{0v} = \frac{1}{2}{\omega_{2\xi}}^2 - {\omega_{\xi}}^3 \tag{47}$$

It is of interest to note that $\mathcal{H}_{0h}$ when substituted in (29) does not give the equation of motion (35). Although (35) can be obtained by using $\mathcal{H}_{0v}$ in (29) expressed in terms of ω.

Conclusion

I conclude by noting that the present work starts with the (1+1) dimensional cylindrical Korteweg-de Vries equation for dust ion acoustic waves in unmagnetised plasma. The cKdV equation is then transformed to KdV equation to construct expressions for corresponding Lagrangian densities which need not be unique. These equivalent Lagrangian densities result in compatible Hamiltonian densities for the theory of Zakharov et al[19,20]. ❐

Reference

1. R. Bharuthram and P.K. Shukla, *Planet. Space Sci*. **40** 973, 1992.
2. I. Popel and M.Y.Yu, *Contrib. Plasma Phys*. **35** 103, 1995.
3. Y. Nakamura and A. Sharma, *Phys. Plasmas* **8** 3921, 2001.
4. H. Washimi and T. Taniuti, *Phys. Rev. Lett*. **17** 996, 1966.
5. S. Ghosh, S. Sarkar, M. Khan and M.R. Gupta, *Phys. Lett*. A **274** 162, 2000.
6. H. Schamel, *J. Plasma Phys*. **9** 377, 1973.
7. S. Watanabe and B. Jiang, *Phys. Fluids* B **5** 409, 1993.
8. E. Infeld and G. Rolands, *Nonlinear waves, Solitons and Chaos*, (Cambridge: Cambridge University Press) 1990.
9. S.K. Ghosh, S.K. Gupta and P. Chatterjee, *Phys. Scr.* **90** 125601, 2015.
10. S. Maxon and J. Viecelli, *Phys. Rev. Lett.* **32** 4, 1974.
11. A.A. Mamun and P.K. Shukla, *Phys. Plasmas* **9** 1468, 2002.
12. B. Sahu and R. Roychoudhury, *Physics of Plasmas* **11** 1947, 2004.
13. B. Sahu and R. Roychoudhury, *Physics of Plasmas* **12** 052106, 2004.
14. B. Sahu and R. Roychoudhury, *Phys. of Plasmas*, **10** 4162, 2003.
15. A. Donato and F. Oliveri, *Applicable Analysis*, **58**(3) 313, 1995.
16. N.H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations* edited by (CRC Press, Ann. Arbor), 1993.
17. P.J. Olver, *Applications of Lie Groups to Differential Equations* (Springer-Verlag, NY), 1993.
18. R. Hirota, *Phys. Rev. Lett.* **27** 1192, 1971.
19. V.E. Zakharov and L.D. Faddeev, *Funct. Anal. Appl.* **5**, 18, 1971.
20. C.S. Gardner, *J. Math. Phys.* **12**, 1548, 1971.
21. R.M. Santilli, *Foundation of Theoretical Mechanics I* (Springer-Verlag, NY) 1984.
22. F. Calogero, *Spectral Transform and Solitons*(North-Holland Pub. Co., NY), Vol. 1, 1982.
23. E.C.G. Sudarshan and N. Mukunda, *Classical dynamics: Amodern perspective* (John Wiley and Sons, NY), 1974.
24. P.A.M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School Monograph series No. 2 (Yeshiva Univ.NY), 1964.
25. Y. Nutku, *J. Math. Phys.* **25**, 2007, 1984.