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ARTICLE

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BIONOMIC ANALYSIS OF A MODIFIED LESLIE - GOWER MODEL WITH HOLLING TYPE - IV FUNCTIONAL RESPONSE AND HOLLING TYPE - II PREDATOR HARVESTING WITH TIME DELAY

JYOTISKA DATTA*, ATISH KUMAR SETHY AND D. RAM PRASAD

The dynamic behaviour of a Leslie-Gower model with a Holling type-IV function response and nonlinear predator harvesting was examined in this study. The maturation delay refers to the predators' ability to harvest prey once they reach a certain age. Qualitative analysis, bifurcation theory, bionomic equilibrium, MSY, and the best harvesting policy have all been used to study the model system. We provide an explanation for the delay-dependent and locally persistent asymptotic nature of the interior equilibrium. Regarding the maturation delay, the age-selective harvesting system experiences a Hopf-bifurcation. A unique control's path has been determined using Pontryagin's maximal principle after environments for the MSY and the existence of bionomic equilibrium have been examined.

Introduction

Salmon fish spends their formative years in rivers before migrating to the sea, where they spend their adult lives and gain the majority of their body mass. Returning to the rivers, they reproduce after reaching maturity. In the rivers, lakes, and streams where they naturally occur, northern pike make excellent game fishPike naturally occur in Alaska to the north and west of the Alaska Range, where they have coexisted peacefully with other native species for countless generations[1]. Because they consume a wide variety of food, including salmon fish in fresh water, pike are generalist type[2] creatures by nature. In addition to supporting one of Alaska's most significant industries, the salmon that return to Alaskan streams and are raised there also support a traditional subsistence way of life in the state's rural areas. State biologists gather a lot of data and statistics regarding the size of salmon commercial fisheries in order to make management decisions. The subsistence use of fish resources is given primary attention in fishery management plans[3]. Similar economic and social phenomena are seen in the Hoogly River and the Sundarban Mangrove environment for the purpose of Hilsa fish harvesting from its coexistence with other lower and higher (generic predator[4]) trophic level species. Fishermen in Hilsa are particularly successful in harvesting fish for both domestic and industrial use[5]. Knowledge of different fishing methods and information about fish behaviour, including migration, foraging, and habitat, are intricately linked. This extra information is frequently necessary for the efficient application of fishing strategies. The target species and its habitat are the major factors that determine whether procedures are appropriate. Age or body size are a species' most important and relevant characteristics since they are correlated with many aspects of its biology, including life history and ecology[6–9]. Fisheries management has fluctuated over the past few decades between alarm and confidence in the direction the management is taking. The strongest scientific concepts for overcoming the worldwide fishing crisis aim to achieve Maximum Sustainable Yield (MSY) via age-selective harvesting, which involves changing gear selectivity, fishing effort, etc.[5,10-12]. Economic and social factors are now included in management objectives in addition to the biological yield of fisheries

* Department of Mathematics, Central University of Odisha, Koraput - 763004 (India)

Corresponding author: jdatta@cuo.ac.in

(MSY)[13–18]. A crucial scientific methodology is proper harvesting, or age/body size selected harvesting, which reduces fluctuations by turning over compensatory dynamics into compensatory dynamics[19–23]. Predator-prey systems are crucial in investigating the dynamics of interacting species, according to various experts[24–27]. Many predator-prey models have been created and examined from diverse perspectives during the past few decades. Because of their universality and significance, the dynamical relationships between predators and their prey have been one of the main issues in mathematical ecology. The dynamical behaviours of predator-prey systems, particularly those with some impulse perturbations, can clearly be affected by predator-prey functional responses at this time. Furthermore, it is common knowledge that time delays play a significant role in mathematical models of ecology. There are typically two forms of time delay in such models: discrete delays and continuous or distributed time delays. In the biological world, there are many instances of processes that require significant delays that cannot be ignored. Examples include the period of time between conception and birth in the case of sexual reproduction, the period of time between the beginning and end of cellular division in the case of mitosis, and the period of time required for digestion before a nutrient is converted into viable biomass. Time delay is crucial to the dynamics of the prey-predator population and has been shown to have a significant impact on whether the prey population is stable or unstable as a result of predation. As delay differential equations exhibit more complex dynamical behaviour than conventional differential equations, delay plays a crucial role in understanding the dynamical behaviour of a system. A stable system can become unstable when there is a temporal delay, according to observations.

In the current work, we have offered a solution for this result. In Section 2, a Leslie-Gower type predator-prey model is used, in which prey grows spatially and is hunted after reaching a certain age gathering. The functional response that Predator uses is the Monod-Haldane[31] or Holling type-IV[30] response. The Michaelis-Menten function, which includes the inhibitory action at high doses in the form of $f(x)=\frac{mx}{\frac{x^2}{i}+x+a}$ where m and a represent, the half-saturation constant and the maximum per capita predator rate in the lack of any inhibiting effects respectively,. The prey-tolerance or immunity of the predator is indicated by the parameter i. Among the three harvesting techniques described in the literature: There are two types of reaping: (i) constant rate, in which a fixed number of people are removed from the population per unit of time; and (ii) proportionate harvesting $h(x) = qEx$, in which the catch is determined by the stock and the amount of effort[32]. where q is referred to as the catchability value, E is the amount of harvesting effort , and qE is referred to as fishing mortality and(iii) nonlinear harvesting, where $h(x)=\frac{qEx}{m_1E+m_2x}$ [32], where m_1 and m_2 are appropriate positive variables and relate to the quantity of stock to the pace of ploughing at high levels of effort, respectively, and to the relationship between the effort level to the rate of harvesting at high stock levels. In this case, we have focused on the third option because it compares the appropriate field conditions. In Section 3, we examined the stability of the delayed model. Simultaneous optimal control strategies have also been investigated in Section 4 for the goals of the bio-economic stability using unsustainable Bionomic equilibrium, the yield and MSY policy, and the best harvesting strategy. In Sections 5 and 6, respectively, the analytical results are followed by numerical examples and a general discussion.

The Model

The following is the Holling type-IV functional response in the Leslie-Gower model[34,35] :

$$\frac{dN}{dT}=rN\left(1-\frac{N}{K}\right)-\frac{mNP}{\frac{N^2}{i}+N+a},$$

$$\frac{dP}{dT}=s\left(1-\frac{nP}{N}\right)P, \qquad (1)$$

with basic circumstances $N(0)\geq 0, P(0)\geq 0$. The population densities of the predator and the prey are shown here as $N\cong N(T)$ and $P\cong P(T)$, respectively. On the set system (1) is defined

$$\Re_+^2=\left\{(N,P)\in R^2 \mid N(0)\geq 0,\ P(0)\geq 0\right\} \qquad (2)$$

And all are nonnegative parameter. Furthermore, Table 1 describes the biological meanings of the characteristics. The Holling type-IV functional response expresses the relationship between both prey and predator[34,36,37]. Predator survival is dependent on prey abundance, hence the conventional ecological sustaining power of the predator, K_p, is assumed to be proportional to prey abundance, U, resulting in $K_p=\frac{N}{n}$ [36]. In the model

system (1), we integrate a nonlinear harvesting $H(N) = \frac{qEN}{(M_1E + m_2N)}$. Thesystem of differential equations[34] is as a result

$$\frac{dN}{dT} = rN\left(1 - \frac{N}{K}\right) - \frac{mNP}{\frac{N^2}{1} + N + a} - \frac{qEN}{(M_1E + m_2N)},$$

$$\frac{dP}{dT} = s\left(1 - \frac{nP}{N}\right)P, \tag{3}$$

with $N(0) = N_0 > 0$ and $P(0) = P_0 > 0$ as beginning conditions. Table 1 provides a description of the parameters. The catchability coefficient and the effort put in to harvest each individual are represented by the parameters q and E, respectively. The constants m_1, m_2 are appropriate parameters.All of the parameters are considered to be positive. On the set R_+^2, the model system (3) is defined. We chose the variables m_1 and m_2 so that $\left(\frac{m_1}{m_2}\right)E < N$. Zhang et al. [33] looked into every aspect of the stability study for non-delayed systems.

Table 1: Biological meaning of the parameters

Parameter	Meaning
r	Prey species' intrinsic rate of development, or the variation between their natural birth and mortality rates
s	Predator species' intrinsic rate of growth, or the difference between their normal birth and mortality rates
k	Prey species' environmental carrying capacity
m	Maximum amount of predation per person
i	Direct evaluation of the predator's sensitivity for or resistance to the prey species
a	Without any inhibitory effects, half saturation remains constant.
n	the quantity of prey needed to maintain one predator in balance
q	coefficient of catchability
E	An effort was made to attract people
m_1	an appropriate positive constant
m_2	an appropriate positive constant
ε	Linear prey fertility rates
μ	linear prey mortality rate
ρ	Prey mortality rate as a result of interspecific rivalry
τ	Late maturation

$$\frac{dN}{dT} = rN\left(1 - \frac{N}{K}\right) - \frac{mNP}{\frac{N^2}{i} + N + a}$$

$$- \frac{q\mu EN(T-\tau)}{m_1\mu Ee^{\mu\tau} + \left\{m_1\rho E\left(e^{\mu\tau} - 1\right) + m_2\mu\right\}N(T-\tau)},$$

$$\frac{dP}{dT} = s\left(1 - \frac{nP}{N}\right)P, \tag{4}$$

If we rephrase the logistic problem in this manner, $\xi = \xi - \mu$, $\rho = \frac{r}{K}$ we get $\frac{dN}{dT} = rN\left(1 - \frac{N}{K}\right) = rN - \frac{rN^2}{K}$

$$= \xi N - \mu N - \frac{rN^2}{K}$$

where r is the intrinsic growth rate, ξ and μ the natural birth and death rates, respectively, and ρ the mortality rate brought on by intra-specific competition. Let us assume

$$N = Ku,\ P = \frac{rKv}{m},\ t = rT,$$

$$h = \frac{q\mu E}{rK\left\{m_1\rho E\left(e^{\mu\tau} - 1\right) + m_2\mu\right\}},\ c$$

$$= \frac{q\mu Em_1e^{\mu\tau}}{K\left\{m_1\rho E\left(e^{\mu\tau} - 1\right) + m_2\mu\right\}},$$

$$\alpha = \frac{i}{K},\ \beta = \frac{nr}{m},\ \gamma = \frac{a}{K}\ and\ \delta\ \frac{s}{r}.$$

The system becomes non-dimensional by using above equations

$$\frac{du}{dt} = u(1-u) - \frac{uv}{\frac{u^2}{\alpha} + u + v} - \frac{hu(t-\tau)}{c + u(t-\tau)} \equiv u\varphi(u, v)\ (say),$$

$$\frac{dv}{dt} = \delta\left(1 - \beta\frac{v}{u}\right)v \equiv v\phi(u, v), \tag{5}$$

With the initial condition

$$u(0) = u_0 > 0 \text{ and } v(0) = v_0 > 0 \tag{6}$$

and all the parameters are non-negative.

Analysis of the System

Positivity and Boundedness of Solution: Using the initial conditions (6), we may integrate Eq. (5) to obtain

$$u(t) = u(0)\exp\left\{\int_0^t\left(1-u-\frac{v}{\frac{u^2}{a}+u+y}-\frac{hu(t-\tau)}{\left(c+u(t-\tau)\right)u}\right)ds\right\}>0$$

$$v(t)=v(0)\exp\left\{\int_0^t\delta\left(1-\beta\frac{v}{u}\right)ds\right\}>0$$

As a result, whenever $x(0) > 0$, $y(0) > 0$ respectively, $x(t) > 0$, $y(t) > 0$ or every $t \geq 0$. Therefore, for the delayed model (5), the interior of the first quadrant is an invariant set.

We also have,

$$\frac{du}{dt}\leq u(1-u)$$

Which gives

$$\lim_{t\to\infty}\sup u(t)\leq 1.$$

We now take a look at a continuous function, $\eta(t) = x(t) + y(t)\ (t - \tau)$. Then

$$\frac{d\eta(t)}{dt}+\mu\eta(t)=u(1+\mu-u)$$

$$-\frac{uv}{\frac{u^2}{\alpha}+u+y}-\frac{hu(t-\tau)}{c+u(t-\tau)}+\left(\delta+\mu\right)v(t-\tau)$$

$$-\delta\beta\frac{v^2(t-\tau)}{u(t-\tau)}\leq\frac{\left(\mu+1\right)^2}{4}+\frac{\left(\delta+\mu\right)^2}{4\delta\beta}=\sigma$$

As a result, according to Birkff and Rota[40], we have $0<\eta(t)<\frac{\sigma}{\mu}\left(1-e^{-\mu t}\right)+\eta(0)e^{-\mu}$ and when $t\to\infty$, $0<\eta(t)<\frac{\sigma}{\mu}$. As a result, $\eta(t)$ is bounded. Consequently, we infer that each solution of the system (5) is bounded by the definition of $\eta(t)$.

Stability of Delayed System: $E\left(u_*, v_*\right)$ is the unique interior steady point for the system (5). Where $v_*=\frac{u_*}{\beta}$ and u_* is the positive roots of $Au^4 + Bu^3 + Cu^2 + Du + f = 0$.

Where

$$A=\frac{1}{\alpha},\ B=\frac{c}{\alpha}+1-\frac{1}{\alpha},\ C=c-\frac{c}{\alpha}-1+\frac{1}{\beta}+\frac{h}{\alpha},$$

$$D=\gamma c-c-\gamma+\frac{c}{\beta}+h\ \text{ and }\ F=\gamma(h-c).$$

Since both of the parameters in this casec and hare delay (τ) dependent, the steady point turns out to be delay (τ) parameter dependent. The model (5) is linearized at $E(u_*, v_*)$, and the result is

$$\frac{du}{dt}=a_{11}u(t)+a_{12}v(t)+b_{11}U(t-\tau)$$

$$\frac{dv}{dt}=a_{21}u(t)+a_{22}v(t) \qquad (7)$$

Where

$$a_{11}=1-2u_*-\frac{\left(\gamma-\frac{u_*^2}{\alpha}\right)}{\left(\frac{u_*^2}{\alpha}+u_*+\gamma\right)^2},\ a_{12}=-\frac{v_*}{\left(\frac{u_*^2}{\alpha}+u_*+\gamma\right)^2},$$

$$a_{21}=\frac{\delta\beta v_*}{u_*^2},\ a_{22}=-\frac{\delta\beta v_*}{u_*},\ b_{11}=-\frac{ch}{\left(c+u_*\right)^2}$$

Given by is the related characteristic equation

$$\lambda^2-\left(a_{11}+a_{22}\right)\lambda+\left(a_{11}a_{22}-a_{12}a_{21}\right)$$

$$+e^{-\lambda\tau}\left(-b_{11}\lambda+b_{11}a_{22}\right)=0 \qquad (8)$$

If and only if, $i\omega(\omega>0)$ is a root of (8), then ω satiefies

$$\left(-\omega^2+\left(a_{11}a_{22}-a_{12}a_{21}\right)-b_{11}\omega\sin\omega t+b_{11}a_{22}\cos\omega t\right)$$

$$+i\left(-\left(a_{11}+a_{22}\right)\omega-b_{11}\omega\cos\omega t-b_{11}a_{22}\sin\omega t\right)=0 \qquad (9)$$

Comparing real and imaginary parts we obtain,

$$\omega^2-\left(a_{11}a_{22}-a_{12}a_{21}\right)=-b_{11}\omega\sin\omega t+b_{11}a_{22}\cos\omega t$$

$$-\left(a_{11}+a_{22}\right)\omega=-b_{11}\omega\cos\omega t-b_{11}a_{22}\sin\omega t \qquad (10)$$

These two equations provide the positive values of and for which the roots of Eq. (8) can be entirely fictitious. When we square and combine these equations, we get

$$\omega^4 + p\omega^2 + q = 0 \tag{11}$$

Where $p = \left(a_{11} + a_{22}\right)^2 - 2\left(a_{11}a_{22} - a_{12}a_{21}\right) - b_{11}^2$ and $q = \left(a_{11}a_{22} - a_{12}a_{21}\right)^2 - \left(b_{11}a_{22}\right)^2$. We will now calculate the value of τ where system (5) will stay stable, then Equation (11) gives us,

$$\tau_k = \frac{1}{\omega_k}\cos^{-1}\left[\frac{\left(b_{11}a_{22} - \left(a_{11} + a_{22}\right)b_{11}\right)\omega_k^2 - b_{11}a_{22}\left(a_{11}a_{22} - a_{12}a_{21}\right)}{\left(b_{11}a_{22}\right)^2 + b_{11}^2\omega_k^2}\right] + \frac{2k\pi}{\omega_k}, \tag{12}$$

Assuming that ω_k is the unique real positive root of equation (11), then k = 0, 1, 2, 3,... Create the function $\theta(\tau) \in [0, 2\pi)$ so that the right-hand sides of the previous equation provide $\cos\theta(\tau)$. The answers to the we seek, at which stability transitions occur, are

$$b_{11}a_{22}(\tau) = \tau - \tau_k. \tag{13}$$

Additionally, by contrasting Eq. (8) with regard to τ, we obtain

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\omega=\omega_0,\ \tau=\tau_0} = \frac{2\omega_0^4 + p\omega_0^2}{b_{11}^2\omega_0^4 + \left(b_{11}a_{22}\right)^2\omega_0^2} > 0,$$

Since $p > 0$ (14)

Theorem: Assuming $\tau > 0$ and the non-delayed ($\tau = 0$) counterpart of system (5) meets the requirements for asymptotic stability, the equilibrium $E(u_*, v_*)$ of system (5) is locally asymptotically stable for $\tau < \tau_0$ and unstable for $\tau > \tau_0$. Additionally, when $\tau = \tau_0$, the system (5) experiences a Hopf-bifurcation at $E(u_*, v_*)$, where

$$\tau_0 = \frac{1}{\omega_0}\cos^{-1}\left[\frac{b_{11}a_{22} - \left(a_{11} + a_{22}\right)b_{11}\omega_0^2 - a_{22}b_{22}\left(a_{11}a_{22} - a_{12}a_{21}\right)}{\left(a_{22}b_{11}\right)^2 + b_{11}^2\omega_0^2}\right].$$

Sustainable Yield and MSY Policy

For a prey species, the maximum sustainable yield (MSY) refers to the biggest yearly catch that can be sustained over time by maintaining the stock at the level causing the most growth.

This output typically changes throughout time depending on what an ecosystem requires to survive. There are numerous restrictions on its yield. The total yield, $V(E^T)$ *t*, in this instance at equilibria $E(u_*, v_*)$ equals

$$V\left(E^T\right) = Eu_* \tag{15}$$

Since *x* is a function of both E and τ. since expressions are highly complex, we will use numbers to illustrate the properties of the total yield $\left(V(E^T)\right)$.

Bio-economic Equilibrium: Economic factors are the aspects of demand and supply that are most significant to the economy. Economic equilibrium happens when the competing forces in the economy are in balance. Unless there is an impact from outside forces on the system, the economic variables will never change. The age-selection technique affects both the biological and economic equilibria differently. We take into variables like to discuss the system's (5) as:

If Cost per unit effort for prey species is equal to $\mathcal{C}$ and $\mathcal{P}$ is the cost per unit of prey biomass.

We have,

$$\mathcal{R} = \left(\frac{\mathcal{P}\mathcal{H}u}{c+u} - \mathcal{C}\right)E,$$

Where

$$\mathcal{H} = \frac{q\mu}{rK\left\{m_1\rho E\left(e^{\mu\tau} - 1\right) + m_2\mu\right\}}$$

The following simultaneous equations produce the bionomic equilibrium $\left(u_\infty, v_\infty, E_\infty\right)$

$$u(1-u) - \frac{uv}{\frac{u^2}{\alpha} + u + \gamma} - \frac{hu}{c+u} = 0$$

$$\delta\left(1 - \beta\frac{v}{u}\right)v = 0$$

$$\left(\frac{\mathcal{P}\mathcal{H}u}{c+u} - \mathcal{C}\right)E = 0 \tag{16}$$

We take into account the following scenarios to determine the bionomic equilibrium:

Case A: Prey harvesting is terminated (E = 0) if the equation $c > \frac{\mathcal{P}\mathcal{H}u}{c+u}$, or the expense is larger than the revenue from the prey.

Case B: If $c < \frac{\mathcal{P}\mathcal{H}u}{c+u}$ the following equation holds, the prey will generate positive income, allowing the system as a whole to function.

In this case $u_\infty = \frac{c\mathcal{C}}{\mathcal{P}\mathcal{H}-\mathcal{C}}$, $V_\infty = \frac{c\mathcal{C}}{\beta(\mathcal{P}\mathcal{H}-\mathcal{C})}$, $E_\infty = \frac{\Omega K m_2 u_\infty}{qrK - \left\{m_1\mu e^{\mu\tau} + m_1\rho K(e^{\mu\tau}-1)u_\infty\right\}\Omega K}$ where $\Omega = 1 - u_\infty - \frac{v_\infty}{\frac{u_\infty^2}{\alpha} + u_\infty + \gamma}$.

Optimal Harvesting Policy: Given that the most desired items are the highest profit with the least amount of effort, policies are one of the most significant difficulties in contemporary society. Clark's[21,32] extensive research covered one of the key aspects of optimal harvest. Ge et al.[41] talk about the best harvesting strategies for stochastic system in 2015. Our goal in this part is to maximize the harvesting system's objective functional form, and the instantaneous yearly rate of discounting ξ is as follows:

$$J(E) = \int_0^\infty e^{-\xi t}\left(\frac{\mathcal{P}\mathcal{H}u}{c+u} - \mathcal{C}\right)Edt \tag{17}$$

subject to the control constraints

$$0 \le E(t) \le E_{\max} \tag{18}$$

and the state constraints (5).

Hamiltonian for the problem is given by

$$H_1 = e^{-\xi t}\left[\frac{PqErKu}{m_1\mu Ee^{\mu\tau} + K\left\{m_1\rho E\left(e^{\mu\tau}-1\right)+m_2\mu\right\}u_\infty}\right] + \in\left[u(1-u) - \frac{uv}{\frac{u^2}{\alpha}+u+\gamma} - \frac{qErKu}{m_1\mu Ee^{\mu\tau} + K\left\{m_1\rho E\left(e^{\mu\tau}-1\right)+m_2\mu\right\}u_\infty}\right] \tag{19}$$

ε is the adjacent variable, in this case

$$\frac{\partial H_1}{\partial E} = \left(e^{-\xi}P - \in\right)\times \frac{m_2 q\mathcal{P}\,\mathrm{r}\,K^2u^2}{\left[m_1\mu Ee^{\mu\tau} + K\left\{m_1\rho E\left(e^{\mu\tau}-1\right)+m_2\mu\right\}u\right]^2} - e^{-\xi}\mathcal{C} = \sigma(t) \tag{20}$$

the ideal regulation E(t) must unambiguously meet the requirements.

$$E(t) = \begin{cases} E^{\max} & if\ \sigma(t) > 0 \\ 0 & f\ \sigma(t) < 0 \end{cases}$$

The reason $E(t)$ are known as switching functions is because they cause $E(t)$ to alternate between level 0 and Emax. The optimal control $E(t)$ is a bang-bang switching from one extreme point to another, depending on the sign of the switching function. The optimal control cannot be established using the above method when the Hamiltonian equation H is independent of the control variable $E(t)$. It is therefore referred to as a solitary control $E(t)$, $0 < E(t) < E_{max}$, and the best harvesting strategy is

$$E(t) = \begin{cases} E^{\max} & if\ \sigma(t) > 0 \\ 0 & if\ \sigma(t) < 0 \\ E^* & if\ \sigma(t) = 0 \end{cases}$$

For singular control $\sigma(t) = 0$. From (a), we get

$$\in = e^{\xi t}\left(\mathcal{P} - \frac{\mathcal{C}\left[m_1\mu Ee^{\mu\tau} + K\left\{m_1\rho E\left(e^{\mu\tau}-1\right)+m_2\mu\right\}u\right]^2}{m_2 q\mathcal{P}\,\mathrm{r}\,K^2u^2}\right) \tag{21}$$

According to the maximum Pontryagain principle[41], the adjoint equation is

$$\frac{d\in}{dt} = -\frac{\partial H_1}{\partial u} \tag{22}$$

From (22) and (19) we have

$$\frac{d\in}{dt} = -\frac{e^{-\xi t}m_1\mu\mathcal{P}qE^2\tau Kue^{\mu\tau}}{\left[m_1\mu Ee^{\mu\tau} + K\left\{m_1\rho E\left(e^{\mu\tau}-1\right)+m_2\mu\right\}u\right]^2}$$

$$-\in\left[1-2u-\frac{\left(\gamma-\frac{u^2}{\alpha}\right)}{\left(\frac{u^2}{\alpha}+u+\gamma\right)^2}\right.$$

$$\left.-\frac{m_1\mu\mathcal{P}qE^2\tau Kue^{\mu\tau}}{\left[m_1\mu Ee^{\mu\tau}+K\left\{m_1\rho E\left(e^{\mu\tau}-1\right)+m_2\mu\right\}u\right]^2}\right] \quad (23)$$

Using the steady conditions (23) ewe get

$$\frac{d\in}{dt}=-\frac{e^{-\xi t}m_1\mu\mathcal{P}qE^2\tau Kue^{\mu\tau}}{\left[m_1\mu Ee^{\mu\tau}+K\left\{m_1\rho E\left(e^{\mu\tau}-1\right)+m_2\mu\right\}u\right]^2}$$

$$-\in\left[-u-\frac{uv\left(\frac{2u}{\alpha}+1\right)}{\left(\frac{u^2}{\alpha}+u+\gamma\right)^2}\right.$$

$$\left.-\frac{qErK^2\left\{uE\rho\left(e^{\mu\tau}-1\right)+m_2\right\}}{\left[m_1\mu Ee^{\mu\tau}+K\left\{m_1\rho E\left(e^{\mu\tau}-1\right)+m_2\mu\right\}u\right]^2}\right] \quad (24)$$

Now putting the values of $\in$ from (21) in (24) we have

$$\frac{d\in}{dt}=-\frac{e^{-\xi t}m_1\mu\mathcal{P}qE^2\tau Kue^{\mu\tau}}{\left[m_1\mu Ee^{\mu\tau}+K\left\{m_1\rho E\left(e^{\mu\tau}-1\right)+m_2\mu\right\}u\right]^2}$$

$$-e^{\xi t}\left(\mathcal{P}-\frac{\mathcal{C}\left[m_1\mu Ee^{\mu\tau}+K\left\{m_1\rho E\left(e^{\mu\tau}-1\right)+m_2\mu\right\}u\right]^2}{m_2q\mathcal{P}\,\mathrm{r}\,K^2u^2}\right)$$

$$\times\left[-u-\frac{uv\left(\frac{2u}{\alpha}+1\right)}{\left(\frac{u^2}{\alpha}+u+\gamma\right)^2}\right.$$

$$\left.-\frac{qErK^2uE\rho\left(e^{\mu\tau}-1\right)+m_2}{\left[m_1\mu Ee^{\mu\tau}+K\left\{m_1\rho E\left(e^{\mu\tau}-1\right)+m_2\mu\right\}u\right]^2}\right] \quad (25)$$

Integrating (25) we get

$$\in=\frac{e^{-\xi t}}{\xi}\left[\frac{m_1\mu\mathcal{P}qE^2rKue^{\mu\tau}}{\left[m_1\mu Ee^{\mu\tau}+K\left\{m_1\rho E\left(e^{\mu\tau}-1\right)+m_2\mu\right\}u\right]^2}\right.$$

$$+\left(\mathcal{P}-\frac{\mathcal{C}\left[m_1\mu Ee^{\mu\tau}+K\left\{m_1\rho E\left(e^{\mu\tau}-1\right)+m_2\mu\right\}u\right]^2}{m_2q\mathcal{P}\,\mathrm{r}\,K^2u^2}\right)$$

$$\times\left[-u-\frac{uv\left(\frac{2u}{\alpha}+1\right)}{\left(\frac{u^2}{\alpha}+u+\gamma\right)^2}\right.$$

$$\left.\left.-\frac{qErK^2uE\rho\left(e^{\mu\tau}-1\right)+m_2}{\left[m_1\mu Ee^{\mu\tau}+K\left\{m_1\rho E\left(e^{\mu\tau}-1\right)+m_2\mu\right\}u\right]^2}\right]\right] \quad (26)$$

Shadow prices are the monetary values attributed to costs that are now unknown or challenging to calculate. The externalization of costs or an unwillingness to bear them are often the causes of these costs to update a system's calculations to incorporate marginal production[42]. Therefore, we disregard the integration constant in order to ensure that the prey species' shadow price, et, is constrained.

From (21) and (26) we get

$$\mathcal{P}-\frac{\mathcal{C}\left[m_1\mu Ee^{\mu\tau}+K\left\{m_1\rho E\left(e^{\mu\tau}-1\right)+m_2\mu\right\}u\right]^2}{m_2q\mathcal{P}\,\mathrm{r}\,K^2u^2}$$

$$=\frac{1}{\xi}\left[\frac{m_1\mu\mathcal{P}qE^2rKue^{\mu\tau}}{\left[m_1\mu Ee^{\mu\tau}+K\left\{m_1\rho E\left(e^{\mu\tau}-1\right)+m_2\mu\right\}u\right]^2}\right.$$

$$+\left(\mathcal{P}-\frac{\mathcal{C}\left[m_1\mu Ee^{\mu\tau}+K\left\{m_1\rho E\left(e^{\mu\tau}-1\right)+m_2\mu\right\}u\right]^2}{m_2q\mathcal{P}rK^2u^2}\right)$$

$$\times\left[-u-\frac{uv\left(\frac{2u}{\alpha}+1\right)}{\left(\frac{u^2}{\alpha}+u+\gamma\right)^2}\right.$$

$$\left.-\frac{qErK^2uE\rho\left(e^{\mu\tau}-1\right)+m_2}{\left[m_1\mu Ee^{\mu\tau}+K\left\{m_1\rho E\left(e^{\mu\tau}-1\right)+m_2\mu\right\}u\right]^2}\right] \quad (27)$$

When steady state equations are solved along with Eq. (27), we obtain the optimal solution $\left(u_\zeta, v_\zeta\right)$ and optimal harvesting effort E_ζ. Eq. (27), which has a unique real positive solution, yields the optimal harvesting effort E^{opt}.

Conclusion

This work proposes a two-species Leslie-Gower type predator-prey model with functional response of Holling type IV, nonlinear prey harvesting, and maturation delay. Modelling the intended delay has been examined using bifurcation concept, Analysis of the biotic equilibrium, the MSY strategy, and the sustainable yield (5). The yield-effort curve for model(5) has been displayed. In Table 2, the stability characteristics of the interior equilibrium point are displayed. There are only two possible stability states: both steady and erratic attention. Model analysis has been done. According to recent research by Zhang et al.[33], the ratio of the intrinsic growth rates of the predator and prey populations and the harvesting rate constant h is the non-delayed counterpart of model(5), which experiences a Hopf- and saddle-node bifurcation around the equilibrium point with respect to the control parameter. Pontryagin's Maximum Principal and an appropriate Hamiltonian function, they have examined the model system's biochemical equilibrium and clarified the best harvesting strategy. In order to ascertain the direction of the Hopf bifurcation and the stability of the periodic solutions that are bifurcating, The spatial version of the model system (5) underwent a detailed stability and Hopf bifurcation investigation by Upadhyay et al.[43]. Numerical techniques have been used to calculate the growth of Turing patterns under nonlinear harvesting. the coexistence of predators and prey populations in space is implied by the stable Turing pattern form, which also has ecological implications because spots patterns are thought to serve as a predator defence mechanism, whereas strips patterns are linked to both social communication and predator defence[44]. ❐

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