DOI: https://doi.org/10.36094/sc.v91.2025.Bhahmaguptas_Kuttaka_and_Chin_Chiu_Shaos.Sinha.479

ARTICLE

Sci. and Cult. 91 (9–10) : 479-487 (2025)

BRAHMAGUPTA'S *KUṬṬAKA* AND CH'IN CHIU-SHAO'S *TA-YEN*

(IN THE CONTEXT OF THE DEVELOPMENT OF LINEAR INDETERMINATE ANALYSIS AND REMAINDER THEOREM)

KANTA PRASAD SINHA

Two neighboring civilisations had been enriching each other since the pre-Buddhist era. The sixth-seventh century's vibrant India with its emancipation of various sheds of Buddhism and mathematics with classical astronomy created deep interest in China. India's astronomers were engaged in the Board of Astronomy in the early Tang Dynasty (618-907 AD).

Verses 32 and 33 of Gaṇitāpada, the second chapter of Āryabhatîya of Āryabhata-1, containing 33 verses in Sanskrit, are the first generalized theory in the History of Mathematics for the solution of Linear Indeterminate Analysis (LIA). The subject became an intellectual stimulant to his able successors; Brahmagupta (598-668 AD) and Bhāskara-1 (600-680 AD), apart other commentators.

The Chinese form of LIA got its shape passing through the Sun Tzǔ suan –ching (280AD-473AD), the fowl's problem (475 AD -500 AD), and slowly development of ch'i-i shu (the Chinese congruence). Before its getting generalized shape in 1247 C E, I-Hsing, (623—727 C E) an erudite Buddhist Chinese astronomer, famous for ta-yen li shu calendarical system, might know the method of indeterminate analysis. Growth in enriching the methods of LIA was very consistent since early medieval India. It had its height in the early twelfth century with Bhaskara-II at its helm, Ch'in Chi'u-shao of China in the thirtieth century concretized the remainder theorem independent of the awe-inspiring Indian tradition of LIA. Before that, the history of Chinese mathematics had no credible evidence in support of a generalized theorem. I-Hsing could have used methods in solving astronomical problems as found in Ch'in Chiu-shao's theorem.

As the focus of this article is presenting the remainder theorem developed six centuries apart, a depiction of the history of LIA for the period from the Sun Tzŭ suan –ching (STs-c) to the final shaping of the ta-yen by Ch'in Chi'u-shao, for China (280 BCE – 1250 CE) and the same for the period of contribution Aryabhata-1, Bhaskara -1 and Brahmagupta for India (500-650 CE) concerning the 'General problem of Remainders' by Brahmgupta are considered for discussion. The contribution of other illustrious successors, Mahavira (c 850 CE), Aryabhata-II (c 950 CE) Sripati (c 1049 CE) and Bhaskara –II (1114-1185 CE) are not felt required.

Introduction

Linear Indeterminate problems were found in most civilisations as a riddle or recreational arithmetic or folk art. Each problem was with a single answer. Probably, problems with uniqueness were set from the empirically chosen units of the answer.

In India, its academic use was found in Baudhayana Śulva Sūtras (c 800-600 BCE) and Apastamba Sulba Sutras (c 500 BCE). Two hundred numbers of bricks were used to form an altar of $7\frac{1}{2}$ a^2 square units with four sizes of

* Retired IAS, Independent Researcher
e-mail: kantoprasad@gmail.com

bricks. Here, one of the answers is: bricks numbers are 24,120,36 and 20 with their respective sizes of each kind of brick are $a^2/16$, $a^2/25$, $a^2/36$ and $a^2/100$ square unit. In Apastamba Sìulba Sütra (c500 BCE), five kinds of 200 number of bricks were arranged for an area of $7\frac{1}{2}$ a^2 square units with 67,58,48, 18 and 9 numbers of bricks with respective size of each brick $a^2/16$ sq. unit , $a^2/25$ sq.unit $a^2/64$ sq.unit $a^2/100$ sq.unit and $a^2/144$ sq.unit. [1-pp 183-185].

In China, The Chiu-chang suan-shu (nine chapters of mathematical techniques) of the first century AD included a problem, a system of four equations with five variables. In its present algebraic form, finding out w, x,y,z,u, and v when w = 2x+y= 3y+2= 4z+u= 5u+v=6v+x. The general solution is w=721t, x= 25t. In the 'The Tzǔ suan ching (Master Sun's Arithmetical Manual, (280-473 BCE) 'there is an unknown number of objects; when counted in "threes," the remainder is 2; when counted in "fives," the remainder is 3; and when counted in "sevens," the remainder is 2. How many objects are there? Answer is 23 [2-p268-269]

The fowl problem of Chang Ch'iu –chien, a contemporary of Āryabhaṭa-I was 'The cock is worth 5 chi'en, a hen 3 chi'en, and 3 chicks in 1 chi'en. With 100 chi'en we buy 100 of them. How many cocks, hens, and chicks are there? Chang provided three answers; (4,18,78), (8,11, 81), (12,4,84).[2-p 277]. Chinese mathematics could not invent generalized theory for centuries. Bhasakara-1 provided four similar examples of residual pulverisers and two examples of non-residual pulverisers following Āryabhaṭa-1's remainder theorem. [3-pp 309-310]

In the Greek anthology (c 500 AD), it is noted 'Three Graces, each carrying the same number of apples in baskets, were approached by 9 muses to distribute the apples. 9 muses and 3 Graces got the same number of apples. What is the number of apples each one got'. The equation reduces to x=4y with 12 n apples. In another such problem, Statue A calls Statue B "What is the weight of mine with the base'. B answers "It is the same with me and my base; (x+y=u+v), then A told that he was twice B's base, B responded that his weight was three times of A. (x= 2v, u=3y), According to Smith, Hindu treatment of indeterminate equations of first degree was original and not influenced by Chinese or Greek writer. [4- p 584]

Āryabhaṭīya of Āryabhaṭa (476-530 AD) being the magnum opus of astronomy and another area of mathematics helped create three types of literature during the period of the next 1000 years;

i) commentaries by Bhaskara -I and others, ii) other Classics of Mathematics over time iii) commentaries of commentaries of predecessors and subsequent Classics.

In the pre-Christian era, 'Nine Chapters in Mathematics' were developed during the 10th -2nd BCE in China. In 1959, Joseph Needham and Wang Ling, noted historians of Chinese mathematics, translated it into English. Alexander Wylie, David Eugene Smith, and Yoshio Mikami called it Arithmetic in the Nine section. Chinese State orthodoxy out of Confucianism viewed the study of Mathematics, Law, and Calligraphy in low social esteem. Those were bracketed for the study of sons of petty officials. [2-p 6]. Mathematicians were used to serve artisans for the practical purpose of technologies in China. The generalised theory of the Indeterminate problem in 'STs-c' took almost a thousand years. Calendar-makers' familiarity with the theory of remainder problem (ta-yen rule) for its use in the method of L*in te li* (665 AD) or *ta-yen Ii* calendrical system (727 AD) was not convincingly proved or disproved.

Āryabhaṭa-1 and Bhāskara-1

To find a number if such is divided by a and b, leaves remainder R_1 and R_2 respectively, in other words, for positive integer solutions of x and y in the equations: N= ax+ R_1 = by+ R_2 were theorized by Āryabhaṭa. [9-p 89]

Bhāskara I, being a successor reposed himself the responsibility in interpreting the obscure and cryptic verses of Āryabhaṭīya in his 'Āryabhaṭīya-bhaṣya'. Besides, he composed Mahā-Bhāskarīya, Laghu-Bhāskarīya. On Āryabhaṭīya, he commented "These aphorisms dealt with the matters beyond the reach of senses." [3- p xxxv].

Some distorted situation in the history made works of Bhaskara-1 mingled with Bhaskar –II, famous for *Līlāvatī* and *Bijagaṇita* of the early twelfth century. In 1930, Datta, B B, an eminent historian of Mathematics drew the attention of scholars regarding independent author, Bhāskara -I, his time and works on astronomy. Books even written in the 18-20th centuries (before 1930AD) had no mentioning of Bhāskara-1 [3-p vii]

His direct disciple Govinda wrote that Bhāskara made an excellent summary of Āryabhaṭīya for 'dull-witted'. Such even was not understood by lesser intellect, so he wrote exposition. Parames′vara added (c 1431) ' ….. therafter Bhāskar wrote a *vṛitti* (of that work) in detail and subsequently Govinda wrote a *bhasya* (comprehensive exposition) of the *vṛiti*. The meaning of that *bhāsya* is not too clear to those with lesser intellect. So, I have written a small expository commentary (of that *bhāsya)* by the grace of God S′iva. [3- pp xxxv-xxxvi]

William Jones (1746-1794) a philologist of merit established the Asiatic Society in Calcutta, Bengal. This noble establishment facilitated the exploration of Indian Mathematics de novo. In the preface of the English translation of 'The Aryabhatia of Aryabhata', W E Clark summarised in 1929 AD that H. Kern edited Āryabhaṭīya, a manual of Astronomy in Sanskrit with its preface, both in English and Sanskrit in 1874. He added Gaṇitāpada, the second chapter of Āryabhaṭīya was translated by L Rodet in the Journal of Asiatique (1879) and George Rusby Kaye's in the Journal of the Asiatic Society of Bengal,1908. [6-preface]

Kaye's who was seen as hardly charitable to India's contribution in Mathematica and astronomy had to admit 'Āryabhaṭa's work contains one of the earliest known to us of an attempt at a general solution of indeterminate of the first degree by the continued fraction process'.[7- p 15] He lamented the past glory of Greece and submitted historical reasons for the failure of Greece in not developing LIA. D E Smith quoted that the method followed was 'continued fractions' based on the findings of Kaye. [8-p 156]. L Rodet writes: "I do not know by what means there became widespread among historians of mathematics the belief that the Indians solved the problem before us by means of continued fractions'. [2-p 365]'. Ulrich Libbrecht (L U) wrote that LIA was based on the substitution process. [2- p 371, footnote]

Connotation of Sanskrit words in verses perplexed translators. W E Clark in his. the book 'Āryabhaṭīya of Āryabhaṭa' (1930) recorded that he followed the closest parallel of verses 32-33 of Ga*nitāpāda* with *Brāhmasphutasiddhānta's* XVIII :3-5 with commentary of Parames´vara (1431 AD). But examples, given by him, did not explain when and how interpolators were added or subtracted. [6-pp42-48]

Clark refuted the claim of Kaye that verses were composed by Āryabhata-II (950 AD). S K Ganguli had to write a separate paper "Notes on Indian Mathematics, a criticism of George Rusby Kaye's interpretation" for Kaye's work on systematically distorted facts. Kaye was known for editing Bakhsali Manuscripts on the sudden demise of noted indologist R Hoernle.

Bitter debate out of a misunderstanding of cryptic Sanskrit verses died down after B B Datta's two sets of translations) following Bhāskara-1 in 1932 C E appeared. Its algebraic interpretation provided full clarity for positive integer solutions of two unknowns, which was reproduced in the History of Hindu Mathematics. [9- pp 94-101]. K S Shukla in collaboration with K V Sarma presented another two sets of translations in 1976. of the same verses. [5-pp 75-78]

Bhāsakara-I: contributed to three major areas for LIA. i) Interpreting cryptic verses in Āryabhaṭīya ii) a generic form for analyzing 'Planetary Pulveriser' and iii) constant pulverise. (*Sthira kuṭṭaka*) in the form of ax ± by=1 in Mahā-Bhāskarīya'. He made larger calculations easier through the theory of constant pulveriser

In planetary motions, the distinct feature of the equation is in the form of $\frac{ax-c}{b}=y$, where 'a ' is the revolution-numbers of a planet (dividend), ie a planet makes revolutions in 432000 years and b (divisor) is civil days in a *yuga* (divisor), a and b being prime to each other, c is the residue of the revolutions of the planet (interpolator), 'x' is *ahargaṇa* (multiplier), days elapsed from the beginning of the yuga and 'y' is a complete revolution performed by the planet (quotient). In other words, if a planet takes b civil days for 'a' rounds in a yuga, then in x civil days, it makes (ax)/b rounds which are equal to y complete revolutions (y) + residual fraction of revolution say, c/b. [10- pp 29-30 and 9 (part II) - p 100]

Bhāskara-I taught in transforming an equation in the form of ax – by = c to ax′ – by′ = 1, then converts to its original form. This compulsion was for handling very large values of c. Let, possible roots be α and β of equation: ax – by = c, then $a\alpha - b\beta$ =c. Dividing both sides by c; the equation turns into $a(\alpha/c) - b\ (\beta/c) = 1$, that is ak–bu=1. $\alpha/c=k$, $\beta/c=u$ Now solving k and u, roots α (c.k) and β (c.u) can be worked out. [10-pp 32-33]

Brahmagupta: Verses in Brahmasphuta siddhānta. (XVIII 3-5,9-11,13-15) are an extended and lucid version of Āryabhaṭīya's LIA with finding the Number, $N = ax + R_1 = by + R_2$, $R_1 - R_2 = c$; (by=ax+c)] is explained. He suggested mutual reciprocal divisions of a and b, being their co-prime, with the number of quotients up to an even number excluding the first one. For finding out the last quotient, the addition of an interpolator with the product of the remainder and the chosen number was the Rule. There is no absolute need in completing the mutual division.

Without considering the first quotient, the chain results $(\overline{x})$, thus obtained, is divided by the divisor corresponding to the smaller remainder. The residue, being multiplied by the divisor corresponding to a greater number and added to the greater number will be the number. For generic form ax–by = ±c; a,b,x, and y are all positive integers. He a,b,x and y are all positive integers. He, being an algebraist transformed equation by = – ax+c causing integer solution with x or y becomes negative. His other major contribution is ' General Problem of Remainders. (GPR) [11 - pp 235-236 and 242].

GPR: Br. SpSiXVIII -55: (XVIII, Rule :55) This is to find a number N which is severally divided by a_1,a_2, a_3, a_4.a_5.................................a_{n-2},a_{n-1}, and leaves remainders as r_1,r_2, r_3, r_4.r_5..........r_{n-2},r_{n-1}, and $r_n$respectively. Form of such equations are $N = a_1x_1+r_1 = a_2x_2+r_2= a_3x_3+r_3 = a_4.x_4+r_4 \ldots\ldots\ldots.. = a_{n-1}x_{n-1}+r_{n-1} = = a_n.x_n+r_n$

Āryabhaṭa-1 provided solutions to these equations with the cryptic word in said verse 33 where in the last line of the translation is as follows' the result is the *dviccheāgraṃ* (द्विच्छेदाग्रम) i.e. the number answering to two divisors). This is also the remainder corresponding to the divisor equal to the product of two divisors'. [1, pp 74-75]

$N=a_1x_1+r_1=a_2x_2+r_2$, now let us take the minimum value of $x_1$as $\alpha_1$which satisfies both equations. Then the general value of $x_1= \alpha_1+a_2.t$. Now, $N=a_1(\alpha_1+a_2.t) + r_1=(a_{1.}\ \alpha_1+r_1) +a_1.a_2.t = a_1.a_2.t+a_1.\alpha_1+ r_1$, where t is an integer, $a_1.\alpha_1+ r_1$ is there remainder left on dividing N by $a_1.a_2$.(9, p-132, part II). If we proceed with similar equations with other variables including remainder, $N= a_1.a_2.t+a_1.\alpha_1+ r_1 = a_3x_3+r_3$, let, t= α_2 be the smallest solution satisfying both the equations, the general value of 't' is $\alpha_2+ a_3.t_1$, then $N=a_1.a_2.(\alpha_2+ a_3.t_1) + (a_1.\ \alpha_1+ r_1) = a_1.a_2.a_3t_1 + (a_1.a_2.\alpha_2 + a_1.\ \alpha_1+ r_1)$

For next equation, $N = a_4.x_4 + r_4 = a_1.a_2.a_3t_1 + (a_1.a_2.\alpha_2 + a_1.\ \alpha_1+r_1)$.

Now, the minimum value of 't_1,' is α_3, then its general value is '$\alpha_3+a_4.t_2$,', putting the general value of t_1, we get $N=a_1.a_2.a_3.\ (\alpha_3+ a_4.t_2) + (a_1.a_2\alpha_2+a_1.\ \alpha_1 +r_1) = a_1.a_2.a_3a_{4.}t_2+ (a_1.a_2.a_3.\ \alpha_3 + a_1.a_2,\alpha_2 + a_1.\ \alpha_1 +r_1) = a_5x_5 + r_5$. The process is continued. till all the equations exhaust.

Pṛthūdakaswami (860 AD) clarified that if p is the LCM of a_1 and a_2, the general value of N satisfies the above two conditions, $N = p.t+ a_1.\ \alpha_1 + r_1$ or, if the common multiple among the numbers is g, then the expression is $[(a_1.\ a_2) /g.].\ t + a_1.\alpha_1.+r_1)$. [(9, p.132 (footnote), part-2]

The contribution of Āryabhaṭa-1-Bhāskara-1-Brahmagupta is enough to match the evolution of *ta-yen* Rule (LIA in China).

I-Hsing (683-727 C E)

I-Hsing was the great Chinese astronomer-mathematician in the Medieval period of China. R C Gupta noted that he was an able mathematician, excelled in astronomy, and became well-versed in Sanskrit. He combined in himself the tradition of Chinese as well as Indian mathematical science. Being a Buddhist monk at an early age, he traveled widely to attend conglomerations of monks and *śramaṇas.*

At the end seventh century and before it, a lot of Indian Mathematicians were known in China The contents of *Chiu-Chih li* were Varahamihira's work on *Surya-Siddhanta*. Compilation of such work in Chinese was done by Gotama Siddha (Quatan Xida). Buddhism reached China in the late Han period (150 C E). Indian astronomy followed it immediately after. Computations with place-value and zero., as developed in India, were formally introduced in China through the literature of contemporary mathematicians-astronomers like Varahamihira, Bhaskara-1, and Brahmagupta. **I-Hsing's** mathematical prowess was found in handling large astronomical calculations with indeterminate problems, 3^{361} and 10^{172}. He was capable of enumerating all possible changes in the chessboard and *go* -–board. [12- p 454]

Tang Court put him in an astro-geodesic survey to refine the calendrical system and gather useful data for solar eclipse prediction. Thirteen test sites from the Baikal Lake in North and Jiaozhou in Vietnam were installed. In Tang Empire those were taken up for monitoring the length of shadows in different periods of the year.

Christopher Cullen in the article "An Eighth Century Chinese Table of Tangents details how I-Hsing developed his table of tangents through indigenised Indian Classical astronomy by using Tangent values, based on Indian Trigonometry. [13 pp. 1-33]

L U and Victor J Katz as well mentioned a problem from I-Hsing's record concerning two indeterminate equations out of revolutions of planetary bodies, with an answer. The present author tried to test if such astronomical data respond to *kuṭṭaka* and the *ta-yen* separately and found that answer attested to both methods.

But, records did not prove or disprove convincingly of his role in shaping the Chinese Remainder Theorem. Chang Tun-jen seems to have thought that I-hsing used Indeterminate analysis. [2-p 282]

Myths surrounding I-Hsing and his works. I-Hsing and I-tsing: I-tsing (635-713 AD), a Tantrik Buddhist monk and scholar came as a pilgrim and spent more than 10 years in Nalanda immediately after the visit of Hiuen Tsan (629-644 AD). Late in the seventh century, he traveled from Canton to Palembang, in the kingdom of Srivijaya, where he spent six months studying Sanskrit in preparation for a longer stay in the Buddhist center of learning at Nalanda in the then Bengal. Indian historians mistakenly mixed up both for their similarities in monkhood, love for Sanskrit knowledge, and study of Indian astronomy and mathematics.

Ta-yen and Ta-yen li: 'li' as a word in the Chinese language connotes 'natural laws' or rites. Its philosophical meaning like ritual or decorum from the time of Confucius gradually changed. *Lin te li, ta yen li* represented the Calendrical system during the early period of the Tang dynasty.

LU mentioned that I-Hsing drew up a calendrical system known as *Ta-yen li shu*, it did not have much to do with indeterminate analysis even though the title is similar. [2-pp 280-281]

I-ching and I-Hsing: IChing: I- Ching is an ancient Chinese divination text on cosmology.

The Shu-shu chiu-chang of Ch′in Chiu-shao contained 81 problems in nine sections. In the first problem of the book, the divination technique of the I-ching, the text was followed. The I-Ching as text' was mixed up with I-Hsing. as a person. This created historical distortion. [2-p 280 and p 293].

L. E Dickson recorded that I –Hsing's t'ai-yen-lei-schu gave a generalization of Rule "Yih-hing proposed to find the number of completed units of work, the same number x of units to be performed by each of four sets of 2, 3, 6, 12 workmen, such that after certain whole days' work, there remain 1, 2, 5, 5 units not completed by the respective sets". [14-p 58]

L U on mistaken attribute to I-Hsing: He noted that (I-Hsing) received attention, mainly because of Matthiesen's error in ascribing to him much of the work of Ch'in Chiu Shao. [2- p-280]. He pointed out the source of the error in interpreting two Chinese words; 't'ai-yen-li-shu' and 'I-Ching'.. Matthiessen attributed I-Hsing without understanding Cbin chi'u Shao's work with the first problem from the I-Ching, the book on divination. LU opined that the contribution of I-Hsing in ta-yen is not well known. [2-p 313, footnote:21]. Matthiessen confused I-ching with I-hsing, both as I-King following Biernatzki and making an incorrect judgment that Ch'in Chiu-shao work was originated from I-hsing Dickson was unaware of this distinction before citing the problem in his book stated before. Matthiessen in an article in 1876 AD corrected the error due to Biernatzki and interpreted the *ta-yen* rule correctly. [2-p 315]

Mathematical tradition in the medieval period of China: Nine chapters of Mathematics (*Chiu- chang *suan -su*), developed during the tenth to second century BCE contained 246 problems. *Chiu- chang *suan -su*, the classic benefitted Chinese reckoning clerks for a millennium. Moreover, post –Confucian mathematics centered on rationalisation and systemization to the architectural standard. Ch'in Chiu-shao and other mathematicians followed the canonical structure of *Chiu- chang *suan -su* with certain variations. *Yu Hai,* encyclopedia of Chinese Mathematics (1267) listed traditional methods of that classic and some old mathematics without mentioning the mathematicians of earlier dynasties.

A family instructions of the Yen-clan of the sixth century instructed members of the family to learn mathematics as an amateur, even though it is considered one of the six arts. Mathematics was not considered a suitable livelihood for a gentleman. 'Office of Mathematics' in the Sui and Tang period was set up to train minor officials. This sort of organisation hindered the growth of mathematics.

No great mathematicians were either students or teachers there. Another establishment to teach a calendrical system was set up in the name of the Board of Astronomy. Ch′in studied at the Board as an exceptional case. A mathematician was considered a technologist; either to solve practical problems out of chronology and astronomy or handle situations out of financial affairs, taxation, architecture, military problems, etc. State orthodoxy of Confucianism disdainfully bracketed law, calligraphy and mathematics to be pursued by sons of petty officials. Ch′in wrote that his work on mathematical problems would serve gentlemen of broad knowledge to pursue in their spare time, for although it was a minor art. [2-pp 4-7]

Chinese civilisation had no school of thought nurtured with high-end scholars similar to 'teachers- disciple students' run generations after generations as was found in Indian Civilisation.

***Kuṭṭaka* or *Ta-yen* in Seventh and Eighth century China**

R C Gupta wrote Chiu-chih li (Nine Upholders Calendrical System) was entirely based on the Indian astronomy of the 7th century. It grew with the tradition of Sûrya-siddhânta, works of Varâhamihira (sixth century) and Brahmgupta. Indian trigonometry based on sines was introduced in China. At the beginning of the eighth century, I-Hsing combined the traditions of both civilisations. [12-p 453]

Matthiessen stated that *kuṭṭaka* is based on continued fractions and *ta-yen* is the method of congruences of Gauss. LU affirmed that *kuṭṭaka* was based on a substitution method. According to Yushkevitch A P (1955), Indian *kuṭṭaka* is different from *Ta-yen* [2-pp 361-362]

The *Ta-yen* calendar and the Huang-chi calendar were initiated by I-Hsing in 724 AD and Liu Ch'o in 600 AD respectively. It is believed that the first one is the origin of the *Ta-yen* rule, the other, with the method of finite differences' [2 p -57] Ch′in Chiu-Shao has mentioned in the preface that calendar-makers used the *Ta-yen* without understanding it. [2 p-. 367]

L U further wrote that data in regard to I-Hsing's calendrical ta-yen method before or after Ch'in is very poor on I-Hsing. Interesting part is that Chinese Mathematicians prior to Ch'in attached too much importance to the STs-c problem, but Ch' in did not mention it, even in preface of his book. [2 p-367 footnote.]. K S Shukla observed that the great extension method **of finding unity** is nothing but congruence. Was I –Hsing, the scholar extra- ordinarily acquainted with the Bhaskara-1's constant pulveriser for its use as $ax \cong 1 \pmod{y}$?

***History of development of Chinese Linear Indeterminate Analysis*.**

The stalwarts of mathematics and astronmy and their accomplished works towards LIA are discussed below.

Two ancient Indeterminate problems in Chinese Civilisation: Surprisingly, post- Ch'in chiu-Shao Chinese Mathematicians nurtured an empirical version of **the STs-c**: problem or the **100- fowl problem** and they were found unaware of Ch'in work.

The STs-c: Even post-Ch'in mathematicians did not lose interest in STs-c. **Yung–Lo ta –tien (Encyclopedia 1407**). The original version is lost. In the problem (stated before), the solution suggested was respective dividend-divisor-remainder as 140-3-2, 63-5- 3, and 30-7-2. Then, adding those dividends and subtracting 210, we get 23, the solution. No explanation was given for the steps.

Yang Hui (1275**)** treated five indeterminate problems of which the first one, 'The STs-c problem with a variant**. He** was able to calculate *fan-yung*. But, did not elaborate on his method. L U commented that his knowledge of LIA was not higher than the process of solving 'The STs-c'.

Chou-Mi: (1290 AD) presented the *ta-yen* rule from the established groundwork of the existing STs-c in the mnemonic stanza, a particular remainder with its particular multiple.

Chou Shu-hsüeh: (1558AD) inaccurately described the STs-c problem.

Cheng Ta –Wei: (1593)'s book 'The Suan fa tung –tsung' included the STs-c problem and was able to give a correct explanation of the STs-c problem, although he failed to understand Ch'in Chiu-shao's method'.

Chang Tun-jen:(1803) first clarified the STs-c with a full explanation.

HsÜ - Ch'un-fang: (active 1935-65 AD) gave a simple explanation of the STs-c problem which might have been the way the Sun Tzu got somewhat mysterious numbers 70, 21 and 15. This particular STs-c problem could easily be conjectured.

STs-c text also contains silly problems; one of such is ' a pregnant woman will give birth in nine months of the year, the gender of the child was asked'.

The 100-fowl problem: **Chang Chiu-chien's** works (468-486AD), if expressed in the algebraic form, its expression is $5x + 3y + \frac{z}{3} = 100$, $x+y+z=100$. No commentators deciphered Chang's rule for his three sets of answers. Chen Luan (570 AD) dealt with a similar problem. Chiao Shun (1763-1820 AD) says the result by Luan was accidental. Liu Hsiao-sun at the end of the sixth century) tried to solve the 100-fowl problem. But the method was similar to that of Chen Luan. Hsieh Ch'a-wei of the eleventh century tried to solve the problem in the same way as Chen Luan.

Ch'iu-i shu: This is connected with solving congruence**.**

Ho Cheng- t'ien: (330-447 AD) a calendar- maker determined the weak ratio and strong ratio lie between the difference between a lunar month and an integral number of days. For two such fractions, with a strong ratio and weak: ratio $\frac{26}{49}$ and $\frac{9}{17}$, the relation was established as $26 \times 17 - 9 \times 49 = 1$. Chen Pao-tsang (1963 AD) opined that ratio $\frac{26}{49}$ came from $\frac{9}{17}$ solving the indeterminate equation, $17x - 9y = 1$. Ho conceived that this form of Ch'iu- i shu is part of congruence in the ta-yen Rule. The question remains whether Ho could develop a fraction into a continued fraction to arrive approximate fraction out of it.

Tsu Chung-Chih (430-501 AD) determined the value of π between two fractions $\frac{355}{113}$ and $\frac{22}{7}$. There are two convergents of continued fractions between these two fractions, there is a relation $(22\times113-7\times355=1)$. Such conjecture of Ch'ien Pao-tsung (1956 AD) was not corroborated by Yushkevitch, AP (1960AD).

Li Shun-feng: (the seventh century**).** Li-devised *Chia tž u yu an'*calendrical system**.** Ch'ien Pao Tsung (1956) opined that he made use of *Ch'iu-ishu* which was put into practice. It is impossible to prove Li's knowledge of the subject.

Shen Kua (1086 AD) mentioned the *Ch'iu-**i** shu* without explanation.

Chi'u– ishu is similar to the **Constant Pulveriser** of Bhaskara-1**.** While the constant pulveriser was focused and designed to assist the calculation of variables, divisors, or dividends for large astronomical data, limited data records are available to establish modest use of **Chi'u – i shu**

Other developments

Lung Shou-i (active 785—803 C E) had written a book '**Suan fa** (Mathematical Methods), but it is lost. **Chang Tun-jen** (1830 AD**)** mentioned the work **of Suan fa.** Li Yen (1940 AD) opined that this could be a work that influenced Ch'in Chiu-Shao. L.U commented that it was a conjecture.

Yen-Kung: (1372AD**)** was the author of the *Tung-yung süan fa*, an alternative manner other than *ta-yen.,* LU observed that Yen-Kung's treatment was very special, not general in nature.

Chou Shu-hsüeh (1558 AD) determined a method of finding *tsung.*

Cheng Ta –wei: (1593). had familiarity with the *ta-yen* which he mentioned as the 'Han Hsin' method. His way of expressing the problem in a mnemotechnical form went viral as recreational mathematics in China and Japan.

Yang Hui (1275 AD) was the biographer of Ch'in Chiu Shao. But neither he nor Chou Mi (1232-1308 AD), an almost contemporary biographer, narrated the STs-c without mentioning Chi'n's *ta-yen* method in his writing.

Ch'in chiu-shao (1202-1261)

George Sarton wrote that Qin was 'one of the greatest mathematicians of his race, of his time, and indeed of all times'. In his youth, as an army commander at Szechwan province in 1226, he was at the forefront to quell the consistent attack of Mongols, even after the death of Gengish Khan (1227 C E). Before that, he studied on the board of astronomy at Hang Chou. He read the Chinese classic 'Nine Chapters of Mathematical Arts'. Besides, being a genius in mathematics, he excelled in writing poems, fencing, archery, and music. He was always in top administrative positions despite his high-end corruption, debauchery, and cruelty. He even poisoned his enemies to death. One of his contemporaries in a letter to the Emperor wrote ' As violent as a tiger or a wolf and as poisonous as a viper or a scorpion'. [15]

Victor J Katz wrote: 'Ch'in Chiu Shao had shaped *ta-yen* rules for linear and higher-order indeterminate solutions. He developed many areas of mathematics with unique solutions in nine chapters of The Shu-shu Chiu-chang. The book included methods of solving equations with higher power like "$x^4 + 763200\,x^2 - 40{,}642560{,}000 = 0$" much ahead of W G Horner. The book also is the oldest extant Chinese Mathematical text with zero symbols; Even then, such mathematical treaties did not make any impact for three centuries in China for the least mathematical activity in Mongol and Ming dynasties. [16 pp 216-221]

An example of the Ta-Yen rule

Before making any effort to apply *ta-yen* principles, let us understand its steps with an example. The example will be helpful to understand the process, the Chinese names of specific stages, and the outcome following each stage. [2- pp 349-355]

If a number is divided by 12, 9,8, and 6 leaves remainders 10, 4,6,4 respectively, but divided by 11,10, and 7, there is no remainder. What is the number?

$N = 10 \pmod{12} = 0 \bmod (11) = 0 \bmod(10) = 4 \bmod (9) = 6 \bmod (8) = 0 \bmod (7) = 4 \pmod{6}$.

Yüan-shu: means problem numbers. 12, 11, 10, 9, 8, 7.6 (m_1, m_2, m_3 m_4 m_5 m_6, .m_7)

ting–shu: Problem numbers are reduced by manipulating common factors among them. Factors of problem numbers are analyzed and set below the problem numbers. This is something manipulative. Factors of LCM are to be put skillfully below the problem numbers.

Here, ***ting –shu*** s are corresponding to problem numbers; 1,11,5,9, 8,7,1, ($\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7$) of 12, 11, 10, 9, 8, 7.6.

yen mu. (L.C.M) When ***ting –shu*** s are multiplied, we find the Lowest Common Multiple in its third stage in this example, it is 1 11 5 9 8 7 1. (27720), expressed as θ. In other words, the second stage is concerned with the placement of factors of LCM just below the problem numbers with certain skills. Factors of LCM on getting their required places (conducive to congruence) become ***ting –shu.*** It seems Chinese mathematics was not comfortable with L C M.

yen-shu. In the fourth stage, LCM (***yen mu***) is divided by a respective factor (***ting shu***).

(***ting shu***). ($M_1 = \theta/\mu_1$, $M_2 = \theta/\mu_2$, $M_3 = \theta/\mu_3$, $M_4 = \theta/\mu_4$, $M_5 = \theta/\mu_5$, $M_6 = \theta/\mu_6$, $M_7 = \theta/\mu_7$ The numbers are 27720/1, 27720/11, 27720/5, 27720/9, 27720/8, 27720/7, 27720/1.

ch'i-shu: At the fifth stage, from the ***yen-shu***, the corresponding ***ting shu*** as many times subtracted; $N_1 = M_1 - p_1 . (\mu_1) = 27220 - 1 \times 127220 = 0$, $N_2 = M_2 - p_2 . (\mu_2) = 2520 - 229 \times 11 = 1$, $N_3 = M_3 - p_3 . (\mu_3) = 5544 - 5 \times 1108 = 4$, $N_4 = M_4 - p_4 . (\mu_4) = 3080 - 342 \times 9 = 2$, $N_5 = M_5 - p_5 . (\mu_5) = 3465 - 433 \times 8 = 1$, $N_6 = M_6 - p_6 . (\mu_6) = 3960 - 565 \times 7 = 5$, $N_7 = M_7 - p_7 . (\mu_7) = 27720 - 1 \times 27720 = 0$

ch'êng-lü : The ***Ta-yen ch'iu-i*** : The stage is very important to find ***ch'êng-lü***.

ch'êng-lü × ***ch'i-shu*** ≅ 1[mod (***ting-shu***)], here seven congruences are $\alpha \times 0 = 1$ (mod1), $\alpha = 0$, $\beta \times 1 = 1$ (mod 11), $\beta = 12$, $\gamma \times 4 = 1$ (mod 5), $\gamma = 4$, $\delta \times 2 = 1$ (mod 9), $\delta = 5$, $\varepsilon \times 1 = 1$ (mod 8), $\varepsilon = 9$, $\zeta \times 5 = 1$ (mod 7), $\zeta = 3$, $\eta \times 0 = 1$ (mod 1) , $\eta = 0$

fan-yung: is the product of ***ch'êng-lü*** and ***yen-shu***, : 0, 12 × 2540, 4 × 5544, 5 × 3080, 9 × 3465, 3 × 3960,0

tsung : is the product of Cheng-yung (***fan-yung***) and remainders. Respective ***fan-yung*** are to be multiplied by remainders (10,0,0,4,6,0,4). These are 0 × 10, 12 × 2540 × 0, 4 × 5544 × 0, 5 × 3080 × 4, 9 × 3465 × 6, 3 × 3960 × 0, 0 × 4. Summation of tsung: $T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 = 0 + 0 + 0 + 61600 + 187110 + 0 + 0 = 248710$. Now, subtracting as much as a multiple of yen mu (LC M) is possible.

Then the desired number is: $\sum$ tsung – nθ = 248710 8 × 27720 = 26950

The rationale of working on the deduction: With moduli relatively prime, $M = m_1 . m_2 . m_3 \ldots\ldots m_n$ $(M/m_1)k_1 \cong 1 \ (mod\ m_1) \cong 0(mod\ m_2) \cong 0\ (mod\ m_3) \ldots\ldots\ldots\ldots\ldots \cong) \ 0\ (mod\ m_n)$, $(M/m_2)k_2 \cong 0\ (\text{mod } m_1) \cong 1(\text{mod } m_2) \cong 0\ (\text{mod } m_3) \ldots\ldots \cong 0\ (\text{mod } m_n)$, $(M/m_n)\ k_n = 1\ (\text{mod } m_n) \cong 0\ (\text{mod } m_1) \cong 0\ (\text{mod } m_2)$, $0 \cong (\text{mod } m_3) \ldots\ldots \cong 0\ (\text{mod } m_{n-1})$

$r_1(M/m_1)k_1 \cong r_1\ (\text{mod } m_1) \cong 0(\text{mod } m_2) \cong 0\ (\text{mod } m_3) \ldots\ldots\ 0\ (\text{mod } m_n)$

$r_2(M/m_2)k_2 \cong 0\ (\text{mod } m_1) \cong r_2(\text{mod } m_2) \cong 0\ (\text{mod } m_3) \ldots\ldots\ldots 0\ (\text{mod } m_n)$

$r_3(M/m_3)k_3 \cong 0\ (\text{mod } m_1) \cong 0(\text{mod } m_2) \cong r_3\ (\text{mod } m_3) \ldots\ldots \cong 0(\text{mod } m_{n-1} \cong 0\ (\text{mod } m_n)$

$r_n\ (M/m_n)\ k_n \cong 0\ (\text{mod } m_1) \cong 0(\text{mod } m_2) \cong 0\ (\text{mod } m_3) \ldots\ldots \cong 0(\text{mod } m_{n-1}) \cong r_n\ (\text{mod } m_n)$ [2 –pp 355-356]

The sum of these tsung taken together satisfies all numbers with respective remainders. $\sum_{i=1}^{n} r_z . (M/m_z). k_z$. The least value is found by deducting the maximum number of LCM from the summation. For getting least value, $\sum_{i=1}^{n} r_i(N_i .).k_i^{-} n. M$ (where N is an integer).

Solving the same problems by application of GPR (BrSpSid. XVIII-55).

1. If STs-c problem. N ≅ 2(mod3) ≅ 3(mod5) ≅ 2 (mod7) is processed with the Brahmagupta's GPR, let the number, is then 3t+2, when t is an integer when divided by 5, leaves 3 as remainder. Then t=2. Number for these two divisors, satisfying the respective remainders is 8.Its general form is 3×5 t_1+8 ; 15t_1+8 when divided by 7, leaves t_1+1 as remainder; here 2 is said as remainder. Then t_1+1 =2 ⇒ t_1=1 , therefore, number is 15×1+8=23
2. The problem which is solved here by *Ta-yen* rule, N ≅ 12(mod 10) ≅ 0 mod (mod 11) ≅ 0 (mod 10) ≅ 4(mod 9 ≅ 6(mod 8) ≅ 0 (mod 7) ≅ 4(mod 6). If it is worked out with above principles, if requires treatment of two numbers at a time when they are relatively prime, if numbers have their common ratios, the product will be divided by the common ratios for LCM of such numbers.
   i) If a number divided by 12 and 11 is divided, the respective remainder is 10 and 0. Then 12t+10, satisfying both the divisors when t

is an integer. If t=1,12×1+10=22, when it is divided by 12 and 11 separately leaves remainders 10 and 0.

ii) Now, finding a number which is divisible by 10 with maintaining previous conditions 12×11 t_2+22= 132 t_1+ 22, is divided by 10, leaves residue '2 t_1 +2' which divisible by 10, then t_1 =4 , number is then 132 × 4 + 22 = 550

iii) If [(12×11×10)/2] t_2+550 is divided by 9, it leaves remainder 3 t_2 +1, if t_2=1, then, 3×1+1 satisfiesits remainder 4; therefore 660 × 1 + 550 = 1210).

iv) Now [(12×11×10×9/(3× 2)] t_3+1210.=1980. t_3+1210 if divided by 8, remainder is 4 t_3+2, if least value of t_3 is 1, then 4 × 1+2 satisfiesits remainder, 6. Then numberis 1980 × 1+1210.=3190 Now, [(12×11×10×9×8) t_4/ (12×2)] + 3190 = 3960 t_4+3190 if divided by 7; 5t_4+5 is to be divisible by 7, then minimum value of t_4 is 6, Hence the desired number is 3960× 6+3190 = 26950.

vi) If 26950. is divided by 6, remainder is 4.

The easier and faster method based on simple iterative process was invented six hundred years ahead of Ch'in's method of congruence.

Conclusion

Calendrical science with its chronological problem required indeterminate analysis for reconciling remainders out of revolutions of planetary bodies. I-Hsing might be conversant with the works of Bhaskara-1 in handling congruences involving large numbers. But there was no systemic record to establish it. The concept of *Chiu-i* (indeterminate analysis) remained undeveloped for centuries in China. Had the constant pulveriser of Bhaskara-1 and Brahmagupta's 'General Problem of Remainders' been known to Chinese Mathematicians, they would use or improved those and they could not struggle for a long period to invent it in the thirteenth century. Despite the inadequacy of more reliable data, the influence of India's LIA including its treatment in astronomical work on I-Hsing could not be ruled out. In a similar vein, it can be concluded that it is not possible to exclude I-Hsing from shaping the *ta-yen* rule in China.

Chin's work stood as the basic form of the Chinese Remainder Theorem in 1247 AD. It received polishing of its rough edges by Beveridge (1669), Euler (1740), and Gauss (1801) in Europe. ❐

References

1. B.B. Datta: Ancient Hindu geometry, Cosmo Publication. New Delhi 1993.
2. Libbrecht Ulrich: Chinese Mathematics in Thirteenth Century. The Shu-shu Chiu-chang of Ch'inChiu-shao, Vol. 1. 1973.
3. Āryabhata: Āryabhaṭīya with the commentary of Bhaskara –I and Someś´vara. Edited by Kripa Shankar Shukla and K.V. Sarma, INSA New Delhi, 1976.
4. D.E. Smith: History of Mathematics. Vol II. Dover Publications Inc. NY., 1931.
5. Āryabhata: Āryabhaṭīya: Edited by Kripa Shankar Shukla in collaboration with K.V. Sarma (A-1) Part -1, 1976.
6. W.E. Clark: Āryabhaṭīya of Aryabhata, (Translation with notes) The University of Chicago Press,1930. Originally published in 1874 by Kern.
7. G.R. Kaye: Indian Mathematics. Thaker, Spink, and Company, 1915.
8. D.E. Smith: History of Mathematics. vol 1. Dover Publications Inc. NY. 1931.
9. B.B. Datta and A.N. Singh: History of Hindu Mathematics. Asia Publishing House; New Delhi, Combined volume (Part-I and Part-II), 1962.

   First published by University of Calcutta, Part -1,1935, Part-II,1958. Asia Publishing House, Combined Volume, Bombay, Calcutta, New Delhi, London, New York. 1962.
10. Bhāskara -1: Mahābhāskarīya, General Editor, Ram Ballav. Lucknow University, India. First Edition. 1960.
11. Brahmahupta: Brāhma-Sputa-Siddhānta, Vol.I Edited by Acharyavara Ram Swarup Sharma, Indian Institute of Astronomical and Sanskrit Research, New Delhi-5, 1966.
12. R.C. Gupta: Gaṇitānanda, Editor: R. Ramasubramanium, Indian Society of History of Mathematics.2015.
13. Christopher Cullen: An Eighth Century Chinese Table of Tangents: Chinese Science, Vol 5 June 82. International Society of East Asian Science, Technology, and Medicine. JSTOR 1982.
14. L.E. Dickson: History of the Theory of Numbers. quoting " L. Matthiessen, Comptes Rendus Paris, 92, 1881, 291; Jour, fur Math", 91, 1881'. Vol II.
15. Mac Tutor History on Qin Jiu Shao by J.J. O'Connor and E.F Robertson., Last updated December 2003.
16. J. Victor Katz: A History of Mathematics, An introduction:3rd Edition: Addison-Wesley Boston San Francisco New York, 2009.