ECONOPHYSICS AND THE KOLKATA PAISE RESTAURANT PROBLEM: MORE IS DIFFERENT

BOAZ TAMIR[1]

We hereby give a broad description and motivation for the new science of Econophysics, in the general scientific context of socio-physics. Next we describe the Kolkata Paise Restaurant Problem (KPR) which is one of the cornerstones of Econophysics. We review the main results of KPR in a simple language. Next we present the notion of quantum games and quantum strategies and thereby quantum KPR. We conclude by suggesting a new version of KPR.

Econophysics: Introduction and Motivations

The term Econophysics was coined by Eugene Stanley during a conference held in Kolkata at 1995, organized by Bikas Chakrabarti. As described by Stanley[1] several years ago, the new discipline had some large amount of data, with no particular laws or theory. Stanley compares this with the state of other disciplines in physics such as superconductance where the phenomena was described way before the presentation of the theory. Econophysics is the unification of two different scientic cultures: " ...a physicist's culture and a typical economist's culture, are really quiet different..". Its biggest challenge as Stanley put it, is to describe fluctuations in economics, and his suggestions to young scientists is to present Big Data into the theory.

Since then the activity grew large, in particular in Kolkata[2]. As Ghosh put it; the main motivation of the Indian Statistical Institute in Kolkata was to promote interdisciplinary research in natural and social sciences. The work on Econophysics started around 1990 in the Saha Institute of Nuclear Physics. To name just a few introductory books on the subject we refer the reader to[3-7].

In Econophysics statistical mechanics is widely used: kinetic theory of gas, percolation, diffusion, self organization, phase transition, etc. Other discipline of physics are also used such as: chaos, network theory, classical and quantum information theory etc. Econophysics relies also on mathematical theories such as game theory, in particular bounded rational potential games.

Econophysics is the new marriage of economy and physics. It is the use of mathematical models used mainly in physics in the description of phenomena known in economics[8]. It is based on the belief that ensemble of people behave like particles, and therefore could be described using method in statistical physics. It is a new and revolutionary form of social science; 'social physics' as will be described below. It was Schelling[9] who was probably the first to use such a method. Schelling was researching the phenomena of segregation by looking at opinions and views of Blacks and Whites. They were asked about their opinion concerning the possibility of living among the others. It turned out that Black people were willing to live among White people as long as they were not the sole Black surrounded by Whites. The same was true for Whites. The results were cast on a computer. It was the time of hype in computer usage describing chaos and fractals. He described a computer game were each pixel corresponds to a person. In the game a black point was moved if it was all surrounded by white points and vise versa. Although the true opinions were very much mild, simulations gave segregations into islands of Blacks and islands of Whites. This means that segregation is bound to

1 Faculty of interdisciplinary studies, Bar Ilan University, Israel, e-mail : boaz tamir@post.bezalel.ac.il

occur even with no racist opinions. It is important to note here that there seams to be no other explanation for the results. Classical sociology could not explain Schelling's results. The emerging 'islands' Schelling observed were an example of an emergent property[10] i.e. a global phenomena which stems out of local relations between the agents. This is the crux of emergent properties, the global properties which emerge are clearly the results of the local interactions, but can not be explained by reduction to those details. This simple research was the cornerstone of a new approach to social science. If we can simulate other phenomena by such 'cellular automata', we can perhaps predict their occurrence and evolution. We can therefore use such cellular automata as a research tool. This is perhaps less than understanding the phenomena, however 'understanding' itself could be very much controversial[11].

In[12] Laughlin suggests the idea that sciences are now in the middle of a revolution, a change from the epoch of reductionism into an age of emergent behavior laws of nature. He also suggests the idea that emergency is in itself a new law of physics. This of course would take 'social physics' very close to other hard-core sciences.

We hereby name just a few examples of socio-physics; in[13] Solomon introduced a mathematical universal diffusion model for two type of agents. The first, *A*, is an enzyme type element that enhance the production of a second element *B*. Simulations show clustering of the *B*s around the *A*s, so the *A*s are like condensation points. In the language of Solomon: Life is discrete! Solomon used his theory to describe several economic phenomena[14]. In[15] Prigogine suggested self-organization to describe urban design. In[16] self-organization was suggested in Game of Life and economics. In[17,18] Havlin was modeling urban growth using percolation theory. In[19] Epstein and Axtell suggested and discussed the notion of 'articial society'. In[20] a master equation for a kinetic model of trading market was discussed. In[21] a new measurement tool was presented to Econophysics: the Complexity- Entropy causality measurement. The new tool was then used to measure stock market ineciency.

The recent abundance of new software such as cellular automata, virtual game software, agent based modeling etc, all bring about a plethora of new results. For a most conclusive introduction see[22], for some particular examples see; simulating diusion of information[23], simulating violence[24], simulating urban behavior[25], simulating opinion formation[6], trac, and even[26,6] trajectories of pedestrians walking in a crowd.

Other organisms like ants, bees, swaps, flock of fishes, all behave in a 'social way' stress the point that we are more similar to simple organisms then we usually think, and some of our social behavior is a natural result of evolution. Looking at flocks of fishes we can identify a process of decision making[27] which is amazingly similar to what we know in humans. It also stresses the fact that some 'social behavior' is indeed an emergent property of local interactions between agents[28,29].

Recently, there is a growing interest in the new science of computational sociology or quantitative socio-dynamics. In[30] Helbing is introducing terms from physics into social science, such as diffusion theory, social forces in terms of the Fokker-Planck equation, gravity models, non linear-dynamics, etc. However, the difference between Helbing's computational sociology and sociophysics as we present here is in the role of the theory of emergent properties.

In the following we will present the main features of this revolutionary idea, its current and future consequences, its meaning in the context of the history and philosophy of science.

The Difference Between Classical Quantitative Social Science and the New Paradigm of Sociophysics : Let us have a quick look at the old paradigm of sociology. In the context of the old paradigm the properties of the group is reflecting the properties of each of its elements. We average the property of the group and relate this average to each of its elements. We also correlate between different properties. One claims that property A causes property B by showing the correlation between the properties, assuming all other variables are the same. We therefore translate correlations into the language of causality. We can also draw a graph of all correlations, thereby getting a path analysis[31].

What then is the difference between our new paradigm and the old one?

The main point in the new paradigm is the stress on the interaction in-between the agents. The properties of the agents themselves are not so important as the interaction in-between, we can thus formulate social sciences in terms of statistical mechanics and thermodynamics[32]. Sociology should be the set of emergent properties (thermodynamics) out of local interactions between neighboring elements (statistical mechanics). Thus if we change the local interactions we will change the global properties. Looking at social science as a theory of emergent properties is revolutionary. It is a bottom up claim, which is well established in other sciences such as neural networks, artificial intelligence, etc. We thereby reduce each of the agents into a set of very simple properties. Many other properties are not relevant. It makes all agents very similar, in fact the same, they therefore loose their identity. This will constitute the main criticism against the theory. However in a theory based on emergent properties there is no place for idiosyncratic values. This new paradigm is a holistic one, in contrast to the old paradigm which is reductive.

This is the content of the proverb 'More is different'[33]. The whole is more than the sum of the parts. Emergent properties are such.

A few words about the term 'Social-Physics'. It was first coined by A.Comte. In 1856 Comte[34] suggested that social sciences should follow hard core sciences in their methods. His new notion of social science was very much positivistic. He described the pyramid of all sciences from mathematics through astronomy, physics, chemistry, biology, then social science. For Comte, social sciences were a natural extension of all other sciences, only growing in complexity. Indeed social scientists adopted some statistical quantitative methods used in the mid 20th century in hard core science, but never followed new methods and new ideas from statistical mechanics, condensed matter physics, etc. E.Durkheim was also suggesting a positivistic, structural and holistic view of the social sciences[35]. Similarly, H.Spencer was suggesting a 'social Darwinism', that is, to look at society from the view point of evolution, as a set of organisms struggling to survive[36].

From Local Interactions to Global Emergency : In[37] Buchanan was inspecting the type of local relations. He showed that one can classify these local relations into several types which he coined: the 'adaptive atom', the 'imitating atom' and the 'cooperating atom'. These three are types of local behavior, or types of strategies. Their emergent properties could explain many social and economic phenomena.

The adaptive behavior is our ability to see patterns and to adjust our response to the observed patterns. Adaptivity is about the dynamics of the local interactions, however the emergent property is a result of our collective adaptivity. A good example is the El-Farol game[38], there Arthur was trying to emphasize the role of inductive reasoning as opposed to deductive reasoning, in particular in cases where there is not enough information or where the situation is complicated and ill defined. In the simple game a group of people are choosing between going to the pub (El Farol) or staying at home. If they went to the bar and it was too crowded they are disappointed. The next time they will use one of several possible strategies to decide whether to go or stay at home. Suppose there is a family of possible strategies from which they can choose. Any player is choosing his strategy independently. Note that everyone knows the number of people that went to the bar the last few times, in other words the players have the same amount of memory. It turns out by simulation that the occupation number will quickly fluctuate around the optimal number - the maximal number of persons the pub can serve without being too crowded. Suppose the number of players is 100, and the optimal number the pub can serve is 60, then the average number of people going to the pub will fluctuate around 60. Moreover several interesting things will happen. First, there will be no clear pattern to the fluctuation, otherwise some of the players will soon use the pattern for their benefit, and then destroy this same pattern by using it. Second, it turns out that exactly 60 percent of the strategies will induce success, (the pub serves less than 60 people) and 40 percent will induce failure (the pub serves more than 60 people). This is due to the fact that the strategies are randomly chosen. The adaptive behavior could also explain the power law distribution of some phenomena, for example in economics it 'explains' (predicts) the appearance of unpredicted events and their respectively high (inverse power law) probability.

Cooperation is well defined in the context of the Tragedy of the Commons[39]. If the group sharing a common property is small and if there are close relations between the agents, then cooperation will sustain. However as the group gets bigger and people get stranger no cooperation will prevail. One solution to the tragedy is to let some third party manage the commons for the benefits of all. Another solution would be by letting the participant join a repeated game, where one is punished in future games for not being cooperative in the current turn. However, the cooperation we get by using such punishments is not natural. Such a cooperation is no more than a sophisticated strategy. Is there a true cooperation between people? It turns out that indeed there is such an intrinsic property in each of us[40] and is deeply connected to the notion of altruism, doing something for the benefits of the whole group with no clear future benefit for oneself. Buchanan talks about true cooperation as a natural one, driven by evolution. This would be a result of natural selection giving a benefit for cooperative and coherent groups over non-cooperative ones.

The imitative behavior: we tend to imitate our neighbors, not as a strategy to gain some payoff but as a primitive property rooted in evolution. This is true in minor things as what we wear, what kind of technology to use, what to buy, and probably also in major things as number of kids to have. In[41] a threshold model was suggested to describe the behavior of individuals, where they join a collective behavior, each having his own threshold, they join in turn when their threshold is crossed. Similarly, in[42] a collective opinion shift was discussed in terms of particles' spin and magnetization. An ensemble of small magnets in low enough temperature will self-organize in the same direction or in a direction of an external close magnet. The magnets will cluster non-continuously to join the whole set. This resembles a formation of an opinion, a fashion change, etc.[37].

We can contrast the above types of local strategies with the general notion of rationality in economics. The assumption of rationality in economics was recently criticized by many. The assumption is most appealing for the construction of analytical models, however in many cases we as human behave non-rationally with respect to our resources[43-46].

The Kolkata Paise Restaurant Problem

The KPR problem is a repeated game defined as follows: each day, a set of agents are looking for restaurants for their lunch. We suppose each restaurant can serve only one agent each day. If more than one agent reaches the same restaurant that day, then one of them is picked randomly. The number of restaurants is N, the number of agents is gN, where g is a parameter of the problem. The next day each agent picks a restaurant according to a set of meta-rules; for example, the agents count the number of agents ni reaching their restaurant i. Then they pick the same restaurant with probability $\frac{1}{n_i}$, or one of the other restaurant with overall probability $1 - \frac{1}{n_i}$, assuming uniform distribution. This means that the agents tend to run away from a crowded restaurant. The KPR problem is a repeated game in the language of game theory, having many versions. In the version we just described the agents, having picked a restaurant decide on the next day restaurant according to its crowding, not looking at other restaurants, however if they decide not to go back to the same restaurant then they pick any other with the same probability no matter where it is located. We can change any of the conditions or add other conditions. In one version, the restaurants are ranked[47], and the agents pick the restaurants according to their rank. Other versions suggest different meta-rules, for example a fixed probability to avoid a restaurant which was over-crowded. In most of the models the probability to pick the next restaurant depends only on the number of agents reaching that restaurant, whether the agent was served there or not.

The KPR problem could be interpreted as resource allocation problem[48]. Suppose a set of computers are given a task to perform as in a case of Big Data. Now the task manager computer has to divide the task to sub-tasks and allocate a free computer to each of the subtasks. Consider also a set of factories and a set of workers looking for a daily job each morning. In case more than one person arrives at the same factory then one is picked randomly (all workers have the same skills). In the old Operations Research theory[49], allocation problems such as transportations were defined to minimize a certain cost function. Those problems were solved using linear programming (LP) methods[50]. In the KPR games discussed below the agents are allocated randomly, there is no analytic solution, therefore we resort to computer simulation and numerical methods. Our goal is to understand the 'emergent properties' such as occupation density, stable states, phase transitions, equilibrium points, etc.

Suppose now we look at the number of unsatisfied agents, the ones that could not be served at any day. We can look at the fraction of them with respect to the overall number of agents. We fine a phase transition between the absorbing state, where all agents are satisfied, and an active state where a fraction of agents are frustrated. Clearly if g is small there are many restaurants and a small number of agents, and therefore shortly enough, after a few steps all agents are satisfied and will stay satisfied. If g is increased then the number of agents gets bigger, some will appear at the same place with high probability, and will start looking for a better place to be served. A fixed fraction of them will stay unsatisfied although changing individual identity. This phase transition is a change in the global behavior of the system (as a function of g).

Let us look at some basic results of the KPR problem, see also[51,52]. The KPR was introduced originally in[53].

Basic Results of the KPR Problem : *Random choice case:* For n players and N restaurants, suppose the probability to pick any of the restaurants is equal to $p = \frac{1}{N}$ then the probability that m players will choose the same restaurant is:

$$\binom{n}{m} p^m (1-p)^{n-m}.$$

For N and n big enough the Binomial distribution becomes Poissonian for $\lambda = \frac{n}{N}$ (use $\lambda = n \, . \, p$) and the above expression becomes:

$$\frac{(n/N)^m}{m!} \exp\left(-n/N\right).$$

Therefore one can compute the probability that a particular restaurant will not be visited by any of the players

(m = 0), and the complementary probability that any number of players will visit this restaurant. Since all restaurants are the same then this last probability will be the average number of occupied restaurants $\bar{f}$ as N and n are big enough. It is easy to see that $\bar{f}$ behaves as a Gaussian function with expectation value around 0.63[53] (use the Poisson distribution above with $n = N$ and $m = 0$ to get the expectation value $1 - e^{-1}$).

Rank Dependent Stochastic: Suppose the k-th restaurant is ranked by k^{ξ} where $\xi \geq 0$, and suppose the probability to pick the k-th restaurant is:

$$p_k = \frac{k^{\xi}}{\sum k^{\xi}}$$

(for $\xi = 0$ we get the uniform distribution). It turns out that the k-th restaurant (for $\xi = 1$) will be occupied with probability $\bar{f}_k = 1 - exp(-2k/N)$ (use $\lambda = n\ .\ p$) and the average over all restaurants is:

$$\bar{f} = \sum \bar{f}_k \Big/ N = 0.57,$$

therefore the average occupation number is smaller when the restaurants are ranked[47].

Strict Crowd Avoiding: The next day all the players (including those that were served the previous day) choose any of the restaurants that nobody went to the previous day. If $\bar{f}$ is the fraction of occupied restaurants at the far future then the next step there are only $N(1 - \bar{f})$ restaurants available, from which they pick randomly. We can now use the above arguments for the random choice case and the condition for being a stationary state to write an equation for $\bar{f}$:

$$\bar{f} = \left(1 - e^{-\left(\frac{n}{N(1-\bar{f})}\right)}\right)\left(1 - \bar{f}\right)$$

It turns out that $\bar{f}$ = 0.46[47].

Stochastic Crowd Avoiding: Suppose now the players are going to the k-th restaurant with probability:

$$p_k = \frac{1}{n_k}$$

where n_k in the number of players that went to the k-th restaurant the previous day, and to all other restaurants with uniform distribution. This means that the players are 'running away' from a previously occupied restaurant with velocity that depends on the amount of occupation. They will run away faster (low probability to stay) if it was highly occupied. By simulation one can show that the average occupation number is a Gaussian with a higher expectation value around 0.8. One can also give an analytical argument proving this expectation value (under the condition that no more than 3 players are visiting the same restaurant, a condition which is indeed shown probabilistically by simulations)[47].

Extended Stochastic Crowd Avoiding: Under the above conditions of the stochastic crowed avoiding, we can extend the case assuming:

$$p_k = \frac{1}{n_k^{\xi}}$$

where ξ is positive. If ξ is low then it will decrease the 'running away' velocity; as ξ goes to 0 simulations show an increase in the utilization function $\bar{f}$, this however also increases the convergence time. At the other end, as ξ goes large, the value of $\bar{f}$ decreases to 0.676[47].

Phase Transition : Now recall that $n = gN$, changing g will change the general behavior of the system. The KPR game shows a (non-equilibrium) phase transition[54,55,56]. Below a certain critical value $g = g_c < 1$ the system will relax on an absorbing state where each of the player finds a restaurant to serve him (for example take a repeated stochastic crowed avoiding game with very few players). Above the critical value there will always be active places i.e. places where more than one player show up in the same restaurant. Let $\rho_a(t)$ be the portion of such restaurants (subscript a for 'active'). Therefore

$$\rho_a(t) \to_{g<g_c} 0$$

$$\rho_a(t) \to_{g<g_c} \rho_a^{\infty}$$

Note that $\rho_a(t)$ is the order parameter of the phase transition. The theory also suggests that near the critical point 0 behaves like $(g - g_c)^{\beta}$ for some $0 < \beta < 1$. Simulations show that indeed this is the case, therefore a true phase transition occurs.

As for the dynamics of $\rho_a(t)$; from the general theory of phase transition[57] we expect that near the critical point g_c, the order parameter $\rho_a(t)$ will be controlled by a scaling function $F\left(\frac{\xi}{\tau}\right)$:

$$\rho_a(t)t^{\alpha} = F\left(\frac{t}{\tau}\right)$$

where τ is the 'relaxation time':

$$\tau = \left(g - g_c\right)^{-\nu_{\parallel}} \sim L^Z$$

L is the size of the system, and $\alpha, \beta\, \nu_{\|}$, and Z are the phase transition critical exponents[57]. For example if $F(\varsigma) = \varsigma^{\alpha}$ then we expect the scaling equation $\nu_{\|}.\alpha = \beta$ to be true.

In case the agents are running away from a restaurant, the next day they can choose any restaurant without any limitation on its distance. This is known as a 'mean field' case. Alternatively, we can assume all restaurants are located on a lattice, and the agents can pick a restaurant only at a nearby location, this is known as the lattice case. In both cases we observe a phase transition as g increases. These phase transitions have order parameters that resemble known order parameters as in sandpile theory[58].

KPR and Minority Games : Suppose there are two ways to drive to my work, A and B. Each morning I have to choose between A and B. Both ways are usually crowded since many of my neighbors, in fact $2M$ neighbors, are working at the same place. We have no way to communicate, however at the end of each day we hear on the radio the traffic details including the number of cars driven on each road. I can use this information to decide which way to drive the next day. Minority games could be looked at as a special case of the KPR problem where there are only two restaurants and each could serve up to M persons. For a general review see also[59]. Suppose at day t road A had $M + \Delta(t) + 1$ cars, and road B had $M - \Delta(t)$ cars. Suppose the agents are using the following stochastic strategy: if at day t they chose the road with the minority of cars (B in the example) then they choose to use the same road at day $t + 1$, otherwise they switch from majority A to minority B with probability p where:

$$p = \frac{\Delta(t)}{M + \Delta(t) + 1}.$$

This is plausible since the running away from the crowded road depends on how much it is overcrowded ($\Delta(t)$). If each of the agents uses the above probability we expect that the average number of people shifting from majority to minority will be:

$$p \cdot \left(M + \Delta(t) + 1\right) = \Delta(t)$$

it seems now that both roads will have about the same number of cars, however there are fluctuations of about $\sqrt{\Delta}$ cars. Hence we go back to the same problem with $\Delta(t+1) = \sqrt{\Delta}$. This means that after about $log(log(N))$ steps $\Delta(t)$ will equal 1, and the next step one of the roads will occupy M cars and the other $M + 1$. This is a frozen state since no agent in the majority road has incentive to switch to the other road.

Avoiding Cheaters : Could one agent cheat? can we guarantee that this will not happen? The cheating agent could stay in majority, hoping that enough agent will switch majority to minority turning his road to the minority one. Similarly he could switch from minority to majority against the rule. He could also switch from majority to minority without using the probability test. We have to fix the parameters of the game so not to allow cheating[60]. It turn out that we can compute the probability of switching between majority to minority for each of the steps, that is for each Δ, so to make the cheating worthless. We suppose all agents know the computation. This is computed under the assumption that there is only one cheating agent.

Continuous Fluctuation : Biswas *et al.*[61] suggested a method that could guarantee steady fluctuations with a continuously controlled amplitude. This could guarantee a way to avoid the freezing of the game ($\Delta = 0$). Suppose each player in the majority set uses the following probability to switch to minority:

$$p_q = \frac{q\Delta(t)}{M + q\Delta(t) + 1}$$

where q is a continuous variable. If $|A| = M + \Delta(t) + 1$ and $|B| = M - \Delta(t)$ then if $S(t)$- the number of agents that switch from A to B equals $2\Delta(t)$ then the same state is formed, only the sizes of A and B swap (almost). We can call it a 'steady state'. It turns out that there is a phase transition around $q = 2$, if the players are using probabilities with $q \leq 2$ the fluctuations will eventually go to 0, and if the players are using higher q then there will be steady states parametrized by q.

Guessing Δ : Suppose each agent can only guess the excess amount Δ, suppose also the guessing is uniform[61], that is the i-th agent guess is:

$$\Delta_i(t) = \Delta(t)(1 + \varepsilon_i)$$

where ε_i is uniformly distributed in $[0, 2x]$. Now each agent uses a different coin with probability:

$$p_i = \frac{\Delta_i(t)}{M + \Delta_i(t) + 1}$$

It turns out that there is a phase transition between an absorbing state and an active state around $x = 1$.

KPR and Social Efficiency: Non-stochastic : Suppose we play the KPR game with N restaurants and N agents, assume that the restaurants are ranked by $0 < k_N \leq k_{N-1} \ldots \leq k_2 \leq k_1$ such that $k_1 \leq 2k_N$. Then the state where each restaurant has exactly one agent is a Nash equilibrium.

This is clear; if agent j having picked restaurant j switches to restaurant i where there is another agent, then his expected payo will be $k_i/2$, now;

$$\frac{k_i}{2} \leq \frac{k_1}{2} \leq k_n \leq k_j$$

therefore his payoff reduces. We could permute the agents between the restaurants, these $N!$ states are all Nash equilibria. Note however that there is no social efficiency, the agent that sits in the highest rank restaurant has the highest payoff. We could rotate the agents non stochastically between the restaurants to solve this problem. Indeed this is done in[62]. The player agree to a cyclic meta-strategy where player j having played strategy j at $t - 1$ will play strategy $j - 1$ at t. There is also a boundary condition; player 1 will play strategy N at time t. Let $\bar{\sigma}$ denote this cyclic strategy. It was then proved in [62] that for $k_1 \leq 2k_N$, the strategy is a Nash equilibrium.

Suppose now one of the player deviate from $\bar{\sigma}$, what can we do? We could punish such players. Consider the following meta strategy for the case of a two restaurant (R1 and R2) cyclic game:

a) Player 1 (2) goes to R1 (R2) at odd games

b) Player 1 (2) goes to R2 (R1) at even games

c) If any of the players deviates, then the other player plays R1 from that point onward.

The condition on k_1 and k_2 makes it a real punishment. For the cases $N = 2$ and $N = 3$ it was proved in[62] that such a cyclic strategy with punishment is an equilibrium point in the phase space of all strategies.

KPR and Social Efficiency: Random Traders : A player that plays randomly between A and B will be called a random trader. Such players can resolve the problem of social efficiency at the freezing point. If $\Delta(t) = 0$ suppose $|A| = M + 1$ and $|B| = M$, then the state is frozen. A random trader that switches between A and B let each of the other players enjoy being in the minority for about half of the time. However the random player himself is always at the majority. We can fix this by taking more random players. Let R be the fraction of random players in M. Consider the steady states discussed above (continuous steady states controlled by a parameter q). We have seen that below $q = 2$ the fluctuations converge to 0, however if there are random trader the fluctuations will be bounded from below. If $R = 1$ the fluctuations will be as large as $\sqrt{M}$. For $q > 2$ where there are non trivial steady states, the random traders will attenuate the fluctuations. This is shown by simulations[61].

Quantum Games : Let us have a look at the Mermin Peres Magic Box game[63,64]. This is a good example showing the ingredients of quantum game theory. Given a 3x3 matrix, a magic box, Alice and Bob are asked to suggest entries under a set of rules. First, each entry is either 1 or -1, each row has an even number of -1's, and each column has an odd number of -1's. Second, Alice and Bob must agree on the entry that is common to both, for example if Alice specify the entries of the second row and Bob specify the entries of the third column, there is one entry (2,3) that is common for both of them and they have to agree on it. Third, each are given an index of a row or a column in advance, we could think of a referee that randomly sends instructions, say the second row to Alice and the 3rd column to Bob. Alice and Bob both win if they specify good entries for that particular pair of instructions. Note that they have no communication. They could agree in advance on a set of strategies but they can not communicate in between the repeated games.

Classically they could act according to a pre-planned Box say:

1	1	1
1	-1	-1
-1	1	?

Given a column and a row they will then specify the above predened entries. However as we can see there is no way to complete the box so to satisfy the rules! Therefore a pre-planning will be good for only 8/9 of the games.

Quantum mechanically however, there is a way to do it with 100% success! Alice and Bob can use a set of entangled states, one for each play. They will measure the entanglement using a set of measurement operators and use the outcomes to specify the entries to the box. It turns out that there is a set of measurement operators and a set of entanglements such that they can always win[65]. What then is the magic?

The rules of the game are local, and could be satisfied by the local quantum scheme. The magic is in the fact that local strategies could be better than global ones. In both the classical and the quantum version of the game there is no communication between the parties, the entanglements produces the correlations between the answers given by Alice and Bob, therefore we have correlations with no communication. There is also the question of complexity. It could be hard to implement the entanglements. Therefore there is the question of finding the simplest entanglement and measurement operators that guarantee the best success probability.

KPR and Quantum Strategies : In game theory it is well known that using quantum strategies can change the landscape of the payoffs[66,67]. Suppose we are given two piles of coins, each coin has a registration number, each coin in the first pile has a corresponding coin in the second pile with the same registration number. If we pick a coin from one of the piles say coin 1 and the corresponding coin from the second pile (with the same registration number 1), then tossing them will give the same results, they both end on Head, or they both end on Tail. These are magic coins, however we can create such coins using the quantum entanglement:

$$\frac{1}{\sqrt{2}}\left(|100\rangle + |11\rangle\right)$$

Both parties measure the entanglement (the particle each holds), then they use the measurement's results to pick the strategy. Using such a process we can force correlations between the pure strategies used by the players. For example if $\{S_1, S_2\}$ are two strategies used by both Alice and Bob, an example of a quantum strategy will be the strategy: a random choice between (S_1, S_1) (Alice and Bob both use S_1) and (S_1, S_1) (Alice and Bob both use S_2). We can force such a strategy by entanglements although we have no communication between the parties. This strategy is not a product of any two strategies and classically would require correlations between the pure strategies. From the point of view of each player, he plays with a mixed strategy (in the language of quantum information theory we use partial trace to describe the strategy used by each of the parties[68]). From the perspective of the payoff matrix it is indeed a special mixed state of the joint strategies. The probabilities and therefore the payoffs will be different than classical.

Quantum KPR : Here we use quantum entanglements to improve the results of resource allocation problems. Suppose we divide an entanglement between the players, suppose also they all cooperate and agree to use this entanglement in their decision of what strategy to use for the next round. We can use the entanglement to force an allocation which is almost optimal. There is no communication between the parties, however we almost write the allocation into the entanglement.

We can play a minority game using entanglements. Suppose an entanglement is distributed among the players. Each will use a local unitary operator on his particle followed by a measurement. The result of the measurement will dene his choice. For example consider the case where there are 4 players. Consider the following entanglement:

$$|\psi\rangle = \frac{1}{\sqrt{8}}\left(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle - |1110\rangle - |1101\rangle - |1011\rangle - |0111\rangle\right)$$

Each player measures his particle in the basis σ_Z. Suppose they all agree that |0> means 'use road *A*' and |1> means 'use road *B*'. Then the entanglement collapses into one of the above states where exactly one player is a winner. For example if the entanglement collapses into |0001> or |1110>, player 1 (the rightmost) will win, all the other will loose. This will occur with probability 1/4. Classically, the probability that player 1 will go to *A* while the other 3 player will go to *B* is $1/2(1/2)^3 = 1/16$, the same probability if player 1 goes to *B*, therefore the probability that player 1 wins is 1/8. Hence the quantum strategy is better. Note that we wrote the correlations needed into the entanglement. We have to assure all players cooperate in using the same measurement process. One also needs to show that any player cannot gain by cheating, that is, by using a different strategy. Note that we are forcing the correlation of strategies without any communication.

In general the players are given an initial entanglement on which they can operate by a unitary operator followed by a measurement. Different unitary operators means different strategies. The problem we face is to find the initial entanglement and the set of unitary operators so to maximize the payoff matrix, above the classical one. In[69] a qu-trit formulation was used to formulate a minority game of 3 players and 3 roads. An initial GHZ type entanglement $\hat{J}|0\rangle^{\otimes 3} = \frac{1}{\sqrt{3}}\left(|000\rangle + |111\rangle + |222\rangle\right)$ was used. The authors presented a unitary operator to be used by all players (written in the 8 dimensional Lie algebra parameters of SU(3)), such that the payoff measurements showed an increase in success probability (up to 6/9) above the classical one (up to 4/9).

A New Version of KPR

In[70] a new version of the KPR game was introduced whereby both sides of the game are ranked. Let X and Y denote two parties, each party rank the elements of the other. The Ys are choosing the Xs by the X rank. Whenever too many Ys choose the same X, then that X will pick some of the Ys by their rank. Here are some motivating examples:

- Consider evolutionary psychology, where each male will search for the 'best fit' female in the neighborhood, and each such female will pick one male that is best fit for her.
- Ph.d. graduates are looking for the best university they could get where the universities will pick the best candidate applying for the job.
- In traffic allocation problems, drivers are looking for the best route from home to job. It could be a toll road and the drivers are paying according to the type of the car, a higher one for a slow or heavy car. In this sense the route 'picks' the best fitted cars.
- We, as clients, choose among several companies for the service we need. The companies will check our credit to make sure we can pay, and will prefer doing business with repeating clients they already trust.

Suppose we have N employers and gN employees. The employees are ranked by their ability to do any work, such as physical condition, age, skills, etc. The factories (employers) are also ranked by their payoffs or social benefits etc. We assume the rank of the factories are well known to all employees. All factories have the same capacity O_c to employ. The game is repeated each day. If several workers are arriving at the same factory (above its capacity) then the owners of the factory pick the best workers by their rank, otherwise they are all employed there. The employees will pick the next day's factory according to their previous day experience. In general, they will surely go back to the same factory if it was under-crowded, otherwise they will go back there with some probability if they were employed there, with less probability if they were refused, and to all other factories by the factories' rank.

We found[70] a phase transition between the absorbing state where all workers find their preferred factories, and an active state where a fraction of the workers are unemployed (frustrated). The phase transition depends on the density g, and occurs around $g = 0.92$. We also studied several correlation properties. The first is a correlation between the rank of the workers and the rank of the factories. Higher ranked workers settle, on average, in higher ranked factories. We also found correlation between the rank of the workers and their idle time, on one hand, and their persistence time on the other hand. Higher ranked workers are less idle and have a larger persistence time. Moreover, we found that increasing the persistence time of the workers by the owners of the factories, i.e., preferring workers they already know, will have global emergent results, with a stronger effect on lower ranked workers.

Many questions were left for future research. What will be the effect of presenting a two dimensional grid of factories and restricting the search for a job over a close neighborhood. It is also plausible to extend the strategy used by the employees to pick the next day employer. For example, one can introduce memory into the game. Employees might remember factories that refused to employ them. Similarly employers might remember employees that left the factory although their rank was satisfactory, letting down the employer. This memory could last for several steps or for ever. Different groups of employees could have different strategy. Some factories could have a lower limit to the rank of workers they can employ. Some factories could have a higher limit, refusing to employ overqualied workers. The ranks could also be dynamic. We can decrease the rank of idle workers, eventually leaving the game (a grand canonical model), the same goes for idle factories. The capacity of the factories could depend on their rank, higher ranked factories could have higher capacity. Some of the workers could have side information on the factories near by, they might change their strategy accordingly etc.

Discussion

Econophysics and in particular the KPR problem suggests many new research directions. We can specify many variants on the strategies used by the agents. The agents can use a different amount of memory and computation power to decide on their strategy. They could cooperate by entanglement, therefore defining a quantum strategy with local unitary operations and operator measurements. Some of the agents can pick a different strategy, therefore there is the question of coalitions. Some may try to cheat therefore forcing the rest to include this possibility in their choice. Some agents may wander randomly. The agents can have a limited amount of information on the behavior of the other agents. The whole game can be played on a lattice therefore forcing the agents to consider only neighboring places. The agents can adapt or cooperate or imitate the behavior of their neighbors. Some agents or all could change their strategies according to the behavior of the whole system. We could also change the parameters of the game, ranking each party, or both. In case there is a lattice we can implement impurities in several places in a form of a different ranking. The parameters of the game could be fine tuned as a result of the evolution of the game, therefore forcing a positive or negative feedback. We can also look at time scales to build a steady state of the system. We can extend our model to a grand canonican one, changing the number of agents from game to game as was done in[71]. All in all we are looking for emergent properties like phase transition from an absorbing state into an active one. We can look at the order parameters of such phase transitions comparing them to those well known in statistical mechanics.

Econophysics suggests the future use of other statistical models as research tools in social science, we name just one, the random sequential adsorption model[72].

References

1. H. E. Stanley, An Interview, IIM Kozhikode Society and *Management Review*, **2**, 2, (2013)
2. A. Ghosh, Econophysics research in India in the last two decades, IIM Kozhikode Society and *Management Review*, **2**, 2, (2013)
3. S. Sinha, A. Chatterjee, A. Chakrabarti, B. K. Chakrabarti, *Econophysics: An introduction, Wiley-VCH*, Berlin, (2010)
4. K. Guhathakurta, B. Bhattacharya, A. Roychowdhury, Examining Stock Markets : a non linear dynamics perspective: Examining the Geometric Brownian Motion model with respect to Stock Price Movement in an Emerging Market, LAP Lambert Academic Publishing, (2012)
5. B. K. Chakrabarti, A. Chakraborti, S. R. Chakravarty, A. Chaterjee, Econophysics of Income and Wealth Distribution, Cambridge University Press, (2013)
6. P. Sen, B. K. Chakrabarti, Sociophysics: An Introduction, Oxford University Press, (2013)
7. F. Abergel, B. Chakrabarti, A. Chakraborti, A. Ghosh, Eds., Econophysics of Systemic Risk and Network Dynamics, Springer-Verlag, (2013)
8. M. H. R. Stanley, L. A. N. Amaral, S. V. Buldyrev, S. Havlin, H. Leschhorn, P. Maass, R. N. Mantegna, M. A. Salinger, H. E. Stanley, Can statistical physics contribute to the science of economics? *Fractals*, **4**, 415, (1996)
9. T. C. Schelling, *Journal of mathematical Sociology,* **I**, 146, (1971)
10. M. A. Bedau, P. Humphreys, Eds., Emergence: Contemporary Readings in Philosophy and Science, A Bradford Book, The MIT Press, (2008)
11. M. S. Morgan, M. Morrison, Eds., Models as Mediators: Perspectives on Natural and Social Science, Cambridge University Press, (1999)
12. R. Laughlin, A Different Universe, Basic Books, (2005)
13. N. M. Shnerb, Y. Louzoun, E. Bettelheim, S. Solomon, The importance of being discrete- life always wins on the surface, arXiv:adap-org/991(2005)
14. M. Levy, H. Levy, S. Solomon, Microscopic Simulation of Financial Markets, *Elsevier*, (2000)
15. I. Prigogine, I. Stengers, Order Out of Chaos: Man's New Dialog with *Nature*, Bantam Books, (1984)
16. B. K. Chakrabarti, S. Marjit, Self-organization in Game of Life and economics, *Indian Journal of Physics B,* **69**, 681, (1995)
17. H. A. Makse, S. Havlin, H. E. Stanley, Modeling urban growth patterns, *Nature*, **377**, 608, (1995)
18. H. A. Makse, J. S. Andrade, M. Batty, S. Havlin, H. A. Stanley, Modeling urban growth patterns with correlated percolation, *Physical Review E*, **58**, 7054, (1998)
19. J. M. Epstein, R. L. Axtell, Growing Artificial Societies: Social Science from the Bottom Up, MIT Press, (1996)
20. A. Chatterjee, B. K. Chakrabarti, R. B. Stinchcombe, Master equation for a kinetic model of trading market and its analytic solution, *Physical Review E,* **72**, 026126, (2005)
21. L. Zunino, M. Zanin, B. M. Tabak, D. Perez, Complexity entropy causality plane: A useful approach to quantify the stock market inefficiency, *Physica A*, **389**, (2010)
22. F. Abergel, H. Aoyama, B. K,Chakrabarti, A. Ghosh, Eds., Econophysics of Agent based models, Springer, (2017)
23. H. R. Nasrinpour, M. R. Friesen, R. D. Macleod, An agent based model of message propagation in the facebook electronic social network, arXiv:1611.07454
24. R. Bhavnani, K. Donnay, D. Miodownik, M. Mor, D. Helbing, Group segregation and human violence, *American Journal of Political Science*, **58,** 1, (2014)
25. O. Scheer, Villes –emergentes, villes mutantes, Ecole National Superieure de Creation Industrielle, (2011)
26. D. Helbing, I. Farkas, T. Vicsek, Freezing by Heating in a Driven Mesoscopic System, *Physical Review Letters*, **84**, 1240, (2000)
27. I. D. Couzin, J. Krause, N. R. Franks, S. A. Levin, Effective leadership and decision making in animal groups on the move, *Nature,* **433,** 513, (2005)
28. A. Berdahl, C. J. Torney, C. C. Ioannou, J. J. Faria, I. D. Couzin, Emergent Sensing of Complex Environments by Mobile Animal Groups, *Science,* **339**, 6119, (2013)
29. I. D. Couzin, C. C. Ioannou, G. Demirel, T. Gross, C. J. Torney, A. Hartnett, L. Conradt, S. A. Levin, N. E. Leonard, Uninformed Individuals Promote Democratic Consensus in Animal Groups, *Science,* **334**, 6062, (2011)
30. D. Helbing, Quantitative sociodynamics, Stochastic Methods and Models of Social Interaction Processes, Springer, (2010)
31. J. L. Moreno, Who Shall Survive, Beacon House, (1953)
32. J. Mimkes, A Thermodynamic Formulation of Social Sciences, in Econophysics and sociophysics, Trends and perspectives, Eds., B. K. Chakrabarti, A. Chakraborti, A. Chatterjee, WILEY-VCH, (2006)
33. P. W. Anderson, More is different: Broken symmetry and the nature of the hierarchical structure of science, *Science,* **177**, 4047, (1972)
34. A. Comte, A General View of Positivism, London 1856, reissued by Cambridge University Press, (2009)
35. A. Giddens, Ed., Emile Durkheim: Selected Writings, Cambridge University Press, (1972)
36. R. Hofstadter, Social Darwinism in American Thought,Beacon Press, (1992)
37. M. Buchanan, The Social Atom, Bloomsbury, (2007)
38. W. B. Arthur, Inductive reasoning and bounded rationality, *American Economics Review,* **84**, 406, (1994)
39. G. Hardin, The tragedy of the commons, *Science,* **162**, 3859, (1968)
40. E. Fehr, U. Fischbacher, The nature of human altruism, *Nature,* **425**, 785, (2003)
41. M. Granovetter, Threshold models of collective behavior, *American Journal of Sociology,* **83,** 142, (1978)
42. Q. Michard, J. P. Bouchaud, Theory of collective opinion shifts: From smooth trends to abrupt swings, *European Physical Journal B,* **47**, 1, (2005)
43. R. Thaler, Winner's Curse, Princeton University Press, (1992)

44. R. Nelson, Economics as Religion, Pennsylvania State University Press, (2001)

45. R. Axelrod, Advancing the Art of Simulation in the Social Sciences, in Simulating Social Phenomena, Eds., R. Conte, R. Hegselmann, P. Terna, Springer (1997)

46. A. Shleifer, Inefficient Markets, Oxford University Press, (2000)

47. A. Ghosh, A. S. Chakrabarti, B. K. Chakrabarti, Kolkata paise restaurant problem in some uniform learning strategy limits, in Econophysics and Economics of Games, Social Choices and Quantitative Techniques, Eds., B. Basu, B. K. Chakrabarti, S. R. Chakravarty, K. Gangopadhyay, New Economic Windows, Springer, (2009)

48. A. Chakraborti, D. Challet, A. Chatterjee, M. Marsili, Y-C. Zhang, B. K. Chakrabarti, Statistical mechanics of competitive resource allocation using agent-based models, *Physics Reports,* **552**, 1, (2005)

49. M. Sasieni, A. Yaspan, L. Friedman, Operations Research: Methods and Problems, John Wiley and Sons, (1959)

50. R. J. Vanderbei, Linear Programming: Foundations and Extensions, Springer-Verlag, (2001)

51. A. Ghosh, A. Chatterjee, M. Mitra, B. K. Chakrabarti, Statistics of the Kolkata paise restaurant problem, *New Journal Physics,* **12**, 075033, (2010)

52. B. K. Chakrabarti, A. Chatterjee, A. Ghosh, S. Mukherjee, B. Tamir, Econophysics of the Kolkata Restaurant Problem and Related Games: Classical and Quantum Strategies for Multi-Agent Multi-Choice Repetative Games, Springer, (2017)

53. A. S. Chakrabarti, B. K. Chakrabarti, A. Chatterjee, M. Mitra, The Kolkata Paise Restaurant problem and resource utilization. *Physica A,* **388**, 2420, (2009)

54. A. Ghosh, D. De-Martino, A. Chatterjee, M. Marsili, B. K. Chakrabarti, Phase transition in crowd dynamics of resource allocation, *Physical Review E,* **85**, 021116, (2012)

55. H. Hinrichsen, Nonequilibrium critical phenomena and phase transition into absorbing states, arXiv:condmat/ 0001070, (2000)

56. S. Lübeck, Universal scaling behaviour of nonequilibrium phase transition, *International Journal of Modern Physics B,* **18**, 3977, (2004)

57. J. C. Toledano, P. Toledano, The Landau Theory of Phase Transitions, *World Scientic Lectures Notes in Physics*, **Vol 3,** (1987)

58. P. Bak, C. Tang, K. Wiesenfeld, Self organized criticality: An explanation of $1/f$ noise, *Physical Review Letters,* **59**, 4, (1987)

59. E. Moro, The Minority Game: an introductory Guide, in Advances in Condensed Matter and Statistical Physics, Eds., E. Korutcheva, R. Cuerno, Nova Science Publishers, (2004)

60. D. Dhar, V. Sadidevan, B. K. Chakrabarti, Cooperation amongst competing agents in minority games, arXiv:1102.4230, (2011)

61. S. Biswas, A. Ghosh, A. Chatterjee, T. Nasker, B. K. Chakrabarti, Continuous transition of social efficiencies in the stochastic strategy minority game, arXiv:1110.5222

62. P. Banerjee, M. Mitra, C. Mukherjee, Kolkata Paise Restaurant Problem and the Cyclically fair norm, in Econophysics of System Risk and Network Dynamics, Eds., F. Abergel, B. K. Chakrabarti, A. Chakraborti, A. Ghosh, Springer-Verlag, (2013)

63. N. D. Mermin, Simple unied form for no-hidden-variable theorems, *Physical Review Letters*, **65**, 3373, (1990)

64. A. Peres, Incompatible results of quantum measurements, *Physics Letter A,* **151**, 3, (1990)

65. P. K. Aravind, Quantum mysteries revisited again, *American Journal of Physics,* **72**, 1303, (2004)

66. J. Eisert, M. Wilkens, M. Lewenstein, Quantum games and quantum strategies, arXiv:quant-ph/9806088, (1999)

67. G. B. Dahl, Quantum strategies, arXiv:1110.4678, (2011)

68. M. A. Nielsen, I. L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, (2000)

69. P. Sharif, H. Heydari, An Introduction to Multi-Player, Multi-Choice Quantum Games: Quantum Minority Games and Kolkata Restaurant Problems, in Econophysics of System Risk and Network Dynamics, Eds., F. Abergel, B. K. Chakrabarti, A. Chakraborti, A. Ghosh, Springer-Verlag, (2013)

70. A. Priel, B. Tamir, Econophysics of a ranked demand and supply resource allocation problem, *Physica A*, **490**, 2018, (2017)

71. T. Galla, G. Mosetti, Y-C. Zhang, Anomalous fluctuations in minority games and related multi-agent models of finantial markets, arXiv:physics/0608091

72. J. W. Evans, Random and cooperative sequential adsorption, *Reviews Modern Physics*, **65**, 1281, (1993)