QUANTUM STATISTICS AND QUANTUM INFORMATION PROCESSING

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Indistinguishability of objects is an abstract intuitive idea that is very difficult to understand from a classical point of view. One can understand identical objects, such as a set of red billiard balls. However, one can easily distinguish these balls one from another by keeping one's eyes on them and following their movement, and so, exchanging their positions or permuting them produces new arrangements. But think of identical objects whose permutations do not produce new arrangements. Such objects, if they exist, would be indistinguishable. They are very difficult to visualize, because it is impossible, in principle, to keep track of them. It is as if their movements occur in a different world not accessible to us. We see nothing like them in our experience. In 1972 one of us (PG) had the great opportunity to attend the 70th birthday celebrations of P. A. M. Dirac at the International Centre for Theoretical Physics in Trieste where Heisenberg was present and gave a lecture on how intuition was more important in physics than abstract mathematical reasoning. At the end of the lecture when everyone was leaving the auditorium, PG found himself close to Heisenberg and plucked up the courage to go near him and ask him what he thought was the main impact of Bose's work on physics of the time. He said, "That particles were indistinguishable, of course, what else?" Text books in quantum mechanics have always emphasized this idea of indistinguishability of photons to be the key intuitive idea that underlies Bose statistics.

In reality, however, indistinguishability of photons did not figure explicitly in Bose's original derivation at all! What Bose actually showed mathematically was the indistinguishability of quantized states of the electromagnetic field having the same number of identical photons. That was a direct *consequence* of quantizing the phase space of the electromagnetic field, not an intuitive conjecture. It *implied*, of course, that photons were indistinguishable.

Bose's starting point was the phase space of the electromagnetic field. Phase space is a six-dimensional space whose axes are labelled by the three components $q_i (i = 1; 2; 3)$ of position and three components p_i of momentum of a system. Each point of phase space thus specifies a 'state' of the system. Planck had proposed that the phase space of the material oscillators that absorb and emit radiation was not continuous but granular with irreducible cells of volume $(\Delta q \Delta p)^3 = h^3$. The number of oscillator states is therefore finite (the states are quantized) and is simply the total volume of phase space divided by h^3. In order to construct a quantized theory of radiation analogous to Planck's quantized theory of material oscillators, Bose proposed that the phase space of the electromagnetic field also had a granular structure, and calculated the *number A* of irreducible cells in phase space that lie within the volume V of a *holraum* with momentum components lying between p and $p + dp$. The result was

$$A = \frac{1}{h^3}\int_V d^3x \int d^3p = \frac{4\pi}{h^3} V\ p^2 dp. \tag{1}$$

Next, Bose used the Einstein relation $p = h\nu/c$ for quanta to obtain the result $A = 4\pi V \nu^2 d\nu / c^3$. It is so simple! But a factor of 2 was missing to get the constant in Planck's law which is $8\pi V \nu^2 d\nu / c^3$. In the published paper one finds the hesitant remark, 'It seems, however,

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appropriate to multiply this number once again by 2 in order to take into account the fact of polarization.'

Calculated in this manner, the first factor acquires a completely new significance:

- It is no longer the number of *modes* of electromagnetic waves in the frequency range lying between ν and $\nu + d\nu$ in unit volume, as is usually shown in text books, but the number of irreducible cells (*or quantized 'states' of light*) in the frequency range $d\nu$ in unit volume.
- The factor of 2 directly indicates that besides the 'orbital degrees of freedom' (i.e. position and momentum), the electromagnetic field posseses an 'internal degree of freedom' which Bose identified with the helicity of its quanta, i.e. the components of its intrinsic spin parallel and anti-parallel to its direction of motion. I will revert to this later.
- Bose wrote in the paper: *The total number of cells must be regarded as the number of possible arrangements of a single quantum in the given volume.*

The rest is just straightforward statistical mechanics. If A^s denotes the number of cells in the frequency range $d\nu_s$ in unit volume, one can easily calculate the number of possible ways to arrange N^s quanta in these cells. Let us place a single quantum of type s randomly in one of the cells, and then a second identical quantum, also randomly, in one of the cells, then a third one, and so on. This way one will get one possible arrangement of N^s identical quanta in the A_s cells, i.e. one 'complexion'. If this procedure is repeated a sufficiently large number of times, all possible complexions will be obtained. Let p_0^s be the number of empty cells, p_1^s the number of cells with 1 quantum, p_2^s the number of cells with 2 quanta, etc obtained in this way. It is clear from this that *the phase cells are distinguished only by their occupation numbers. Hence, cells with a given number of quanta are indistinguishable*. In other words, the different arrangements of the quanta in cells with a given number of quanta are indistinguishable. Since all complexions are equally probable, the probability of a state with type s quanta is

$$W^s = \frac{A^s!}{p_0^s!\,p_1^s!\,p_2^s!\ldots} \tag{2}$$

with

$$A^s = \frac{8\pi\nu^{s2}d\nu^s}{c^3} \tag{3}$$

and

$$N^s = 0.p_0^s + 1.p_1^s + 2.p_2^s + \ldots \tag{4}$$

The macroscopically defined probability of a state defined by all types of quanta is then

$$W = \Pi_s W^s = \Pi_s \frac{A^s!}{p_0^s!\,p_1^s!\,p_2^s!\ldots}. \tag{5}$$

If one maximizes the entropy $S = -k\log W$ subject to the constraints that the total energy is the sum of the energies of all the quanta, $E = \sum_s N^s h\nu^s$, and the total number of quanta is the sum of the numbers of quanta in all the states, $N^s = \sum_s r p_r^s$, one obtains in the limit of large numbers p_r^s the statistical Planck factor $\left(1 - e^{-h\nu/kT}\right)^{-1}$.

We thus see that indistinguishability of photons was not an *ad hoc* assumption in Bose's derivation of the Planck law. It followed from the quantization of the electromagnetic phase space. This is never explained in text books. Bose's derivation shows clearly that *it is not the photons but the quantized states of the electromagnetic field that are the independent degrees of freedom of the system*, and that states with the same number of identical photons are indistinguishable. It is the crucial clue to the Fock space formulation of quantum electrodynamics that followed later.

One more point needs to be mentioned at this stage. If one retains the classical significance of the first factor in the Planck law as the number of modes of the electromagnetic field in a hohlraum, then Planck's law can be deduced if the probability distribution is

$$W = \Pi_\nu \frac{\left(A_\nu + N_\nu d\nu\right)}{A_\nu!\,N_\nu d\nu!} \tag{6}$$

with $A_\nu = 8\pi\nu^2/c^3$. This was first shown by Debye in 1910. In 1911 Natansson[1] and in 1914 Ehren fest and Kamerlingh Onnes[2] showed that the Debye distribution implied the indistinguishability of the quanta.

Hence, there are two ways of going about deriving Planck's law. One can derive the indistinguishability of quantum states of radiation by quantizing the electromagnetic field as Bose did and use the distribution[5], or retain the classical wave character of the electromagnetic field, postulate the indistinguishability of quanta and use the Debye distribution[6]. It is the latter approach which text books have followed.

That is probably because when Einstein extended Bose's method to develop the quantum theory of ideal gases (Bose gases), he used the Debye form of the distribution with the additional assumption of conservation of the number of atoms. Since the gas atoms were indistinguishable by hypothesis, their tracks under mutual exchange became unobservable, leading to their wave nature. Einstein connected these matter waves with de Broglie waves with wavelength $\lambda = h/p$ where the momentum $p = \sqrt{2mkT}$. Hence, $\lambda = \frac{1}{\sqrt{T}}$, and the wavelength was predicted to increase as the temperature of the gas was lowered. Consequently, he showed that the waves of the individual atoms would start to overlap more and more as the temperature was lowered, so that below a very low critical temperature, the waves would overlap so much that the atoms would lose their identities completely and condense into the lowest quantum state, behaving like a giant atom with a wave character. Such a state came to be known as the Bose-Einstein Condensate (BEC).

Ehrenfest was quick to point out that the basic entities that enter into a statistical counting method must be statistically independent of one another, but the method of counting used by Einstein (the Debye distribution) implied that the gas atoms in Einstein's theory were mysteriously correlated. That was a conceptual contradiction. Einstein gave a very perceptive reply to this in his second and third papers on the subject, saying that the method used was the only way of counting that was consistent with the third law of thermodynamics (an extension of Nernst's heat theorem due to Planck) which requires the entropy of all systems to vanish at the absolute zero of temperature, because at very low temperatures all atoms of a Bose gas condense to a single state, and so $W = 1$ and $S = -k\ln W = 0$. No other statistics would give this, and hence the Bose method had to be accepted as correct[3]. Additionally, Planck found the analogy with radiation a weak point of Einstein's gas theory, the statistics of radiation and that of material particles being completely different in his opinion. In his 1925 paper[4] Einstein wrote, "The result presents in itself a support of the view concerning the deep natural relation between radiation and gas, since the same statistical treatment which leads to Planck's formula establishes–when applied to gases–the agreement with Nernst's theorem." Einstein was of the opinion that the mysterious correlations among the gas atoms would have to wait for a better understanding in the future. It should be noted here that the conceptual contradiction pointed out by Ehrenfest does not arise if the original Bose method is used because the independent statistical elements there are not the particles but the quantized states.

The loss of identity of the particles in a BEC was the first clue to macroscopic coherence and entanglement. Entanglement is a key resource in quantum computing, but one of the main problems in making a quantum computer is overcoming decoherence due to the noise on the quantum state of the register caused by the environment. In this respect BECs offer a unique advantage: quantum effects are visible and fairly stable on a macroscopic level. Concrete proposals have already been made to use BECs for quantum computing[5,6].

The key ideas that emerged from Bose's paper were (a) the indistinguishability of quantum states under permutations or exchange of the quanta (eqn (5)) and (b) the existence of 'internal' degrees of freedom of the electromagnetic field (polarization or helicity) due to the additional factor of 2 already referred to above. According to personal reminiscences of some of Bose's students and of C. V. Raman himself, Bose had in mind that this factor of 2 was due to the spinning of the photon parallel or anti-parallel to its direction of motion (helicity). An intrinsic spin of the photon is different from polarization because the latter is state dependent whereas the former is not. Raman and Bhagavantam, in fact, carried out a series of experiments to confirm that Bose was correct. They published their results in a series of papers[7].

Let the internal states of a Bose quantum be treated as a 2-dimensional Hilbert space $\mathcal{H}_s$. In modern terminology it is a qubit. Let there be two Bose quanta. Then, according to the Schmidt decomposition theorem (which predates quantum mechanics!)[8], there always exist states

$$|\psi\rangle_{12} = \sum_{i.j} c_{ij} |i\rangle_1 |j\rangle_2 \tag{7}$$

where $\{|i\rangle_1\}$ is a basis in $\mathcal{H}_{s1}$ and $\{|j\rangle_2\}$ a basis in $\mathcal{H}_{s2}$. This is an entangled state of the two quanta. Hence, entanglement is inherent in Bose's quantization method. A common method of producing entangled states in quantum information processing is to use two qubits and perform a Hadamard operation on the upper qubit and then a CNOT operation with both qubits. The result is an EPR state.

The concept of indistinguishability and quantum statistics, both pioneered by Bose, have entered into the very foundations of quantum mechanics. It is but natural that they should be exploited for quantum information processing. Quantum information processing relies primarily on the creation and manipulation of entangled states of elementary particles and atoms[9]. Among different kinds of elementary particles that are used nowadays in quantum information experiments, such as photons, electrons, neutrons, ions, atoms, etc., photons constitute the most widely used examples. Producing entangled photons is thus of foremost importance in quantum information processing experiments. One of the most popular methods of preparing entangled photons is through the process of parametric down conversion[10]. The generation of an entangled pair of photons when a high energy photon hits a non-linear crystal relies solely on the indistinguisability of the two photons at the output[11]. If instead of two photons, any other kinds of distinguishable particles were emitted, entanglement among them would not be created unless they were made to interact with each other. Thus, the concept of indistinguishability of bosons has indeed played a central role in the modern advancement of quantum information science.

Further investigations on the role of indistinguishablity in quantum information processing have been made in recent years[11,12,13,14,15]. Though it is widely appreciated that entanglement plays a central role in quantum communication between macroscopically separated parties, the characterization of quantum correlations at short distances (where the role of quantum statistics becomes important) is still an open problem[16]. Symmetrization (anti-symmetrizations) of states of two-boson (fermion) systems leads to the familiar mathematical structure of entanglement for two-qubits, for example. Though it is difficult to utilize this kind of entanglement as a resource for performing typical information processing tasks such as teleportation, dense coding, secret key generation etc., unless the particles are well-separated out to make them distinguishable, it has been observed that indistinguishability may act as a kind of utilizable feature in certain quantum enhanced phenomena[14]. Entanglement concentration[17] and state discrimination[18] using quantum statistics have been already proposed.

The question as to what extent a pair of two identical particles need to be spatially separated in order to utilize them in a typical quantum information theoretic task such a teleportation, remains open, and may lead to certain paradoxical situations. For example, consider the three-qubit Greenberger-Horne-Zeilinger (GHZ) state [19] which is a highly entangled state of three particles. When one of the particles is traced out (lost or not observed), the resultant state of two particles is separable with no entanglement. Now, if we consider a physical situation of a GHZ state composed of three identical particles from which one of the particles is taken far away and the other two brought closer together in space, what would be the outcome? Will one get a two-qubit separable state (effectively ignoring the third particle) or a two-qubit maximally entangled state (as a consequence of (anti-) symmetrization of the two-particle wave function due to the spatial overlap when they are brought close together)? The structure of entanglement is so far thought to depend only upon the Hilbert space geometry[9]. However, the above situation forces us to consider spatial geometry in addition to Hilbert space geometry for determining entanglement of indistinguishable particles. Such questions may indeed acquire practical relevance with modern experiments demonstrating GHZ states with Rydberg atoms[20].

An important concept in quantum information processing is entanglement quantication. This is done fundamentally through the von Neumann entropy of the states. All entanglement measures are based on some quantity or the other directly related to the von Neumann entropy. However, this is not satisfactory for identical particles[15]. Also, we have seen that the Boltzmann entropy of the lowest state in a Bose gas (an entangled state) vanishes in accordance with the third law of thermodynamics which is universally valid, i.e. true for all systems. The third law provides an absolute reference point for the determination of entropy at any other temperature. Interestingly, the von Neumann entropy vanishes only for pure systems. This area of mixed and pure states, temperature, entropy, the third law and their relevance for quantum information processing, if any, remain largely unexplored so far, and is probably worth following up. Recently, operator-based methods have been developed that attempt to understand the structure of entanglement of identical particles in many-particle systems[21]. However, much more work needs to be done in order to probe quantum correlations in realistic scenarios where particle statistics are relevant, such as in the physics of Bose-Einstein condensates, quantum dots and biological molecular aggregates. Progress on outstanding fundamental problems such as the black-hole information paradox[22] requires a clear formulation amalgamating Bose's indistinguishability and quantum statistics in the framework of quantum information.

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