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Application of Leith's Diffusion Approximation to Homogeneous Isotropic Turbulence

***Abstract* : L. F. Richardson[1] (1926) considered first that small-scale disturbances in a turbulent flow with sufficiently large Reynolds number may reduce to an isotropic turbulence state. The idea was extended by A N. Kolmogorov[2] (1941) who introduced turbulence energy cascade and proposed several similarity hypotheses at sufficiently high Reynolds number. Kolmogorov termed this turbulence which comprises of various length and velocity scales as "locally isotropic turbulence".**

In this paper, we shall apply a general type of self-similar scheme for the decaying homogeneous isotropic turbulence under the condition that Reynold's number is very high. It is shown here that, the diffusion approximation due to Leith[3] (1967) can be applied straightforwardly and Kolmogorof's 5/3rd law is obeyed. Applicability of this problem is discussed with respect to similar other models. We would limit our attention here closely to the Leith's model of evergy transfer only following the thoughts of Richardson-Kolmogorov cascade and equilibrium hypothesis.

Key Words : Homogeneous and Isotropic Turbulence, Diffusion Approximation, Similarity solution.

Leith presented a model depicting a diffusion approximation in the non-local inertial energy transfer between the wave number components in the spectral representation of a homogeneous isotropic turbulent flow. According to Connaughton and Nazarenko[4] explained that Leith's model on turbulence may actually be represented by a non-linear degenerate diffusion equation. These authors constructed a turbulence model such that in the case of vanishing viscosity, there are two steady state solutions namely (1) Kolmogorof Spectrum that corresponds to the cascade state and a thermodynamic equilibrium distribution. Further, in a later paper, S. Thalabard[5] et al. predicted an analytical expression for the anomalous exponent of the transient spectrum and description of formation of Kolmogorof type spectrum as a reflection wave from the dissipative scale back into the inertial range. In the present paper. We would like to investigate the nature of a most general type of homogeneous and isotropic energy spectrum following an earlier model presented by N. R. Sen[6,7,8].

Formulation of the Problem : The decay equation of the energy spectrum $E(k,t)$ in homogeneous and isotropic turbulence under the energy transfer following Leith's model may be written as

$$\frac{\partial}{\partial t}\int_0^k E(k',t)dk' = -W(k,t) - 2\nu\int_0^k k'^2 E(k',t)\,dk' \quad (1)$$

where $W(k,t) = -2\gamma_L k^{\frac{13}{2}}\frac{\partial}{\partial k}\left(k^{-3}(E(k))^{3/2}\right)$, γ_L being a dimensionless constant.

Solution of the Problem : We assume a general solution of (1) in the form, after Sen (1951) as

$$E(k',t) = \frac{1}{\gamma_L^2 k_0^3 t_0^2}\frac{s^3}{\tau^2} f\left(\frac{sk'}{k_0}\right) \quad (2)$$

where $s = s(\tau)$, $\tau = \frac{t}{t_0}$ and k_0, t_0 and γ_L are constants. Substituting (2) on the left hand side of (1), we get

$$-\frac{\partial}{\partial t}\int_0^k E(k',t)dk'$$

$$= \frac{s^2}{t^3}\frac{1}{\gamma_L^2 k_0^2}\left[\int_0^x 2\left(1-\frac{\tau}{s}s_\tau\right) f(x')dx' - xs_\tau\frac{\tau}{s}f(x)\right] \quad (3)$$

where $s_\tau = \frac{ds}{d\tau}$

Now, the RHS of (1) (with proper care for Sign)

$$= W(k,t) + 2\nu\int_0^k k'^2\, E(k',t)\,dk'$$

where $W(k,t) = 2\gamma_L k^{\frac{13}{2}}\frac{\partial}{\partial k}\left(k^{-3}E(k)^{3/2}\right)$

$$= 2\gamma_L k^{\frac{13}{2}}\frac{\partial}{\partial k}\left(k^{-3}\frac{1}{\gamma_L^3 k_0^{9/2} t_0^3}\frac{s^{9/2}}{\tau^3} f^{3/2}\left(\frac{sk}{k_0}\right)\right)$$

$$= 2\gamma_L k^{\frac{13}{2}} \times$$

$$\left[-3k^{-4}\frac{1}{\gamma_L^3 k_0^{9/2} t_0^3}\frac{s^{9/2}}{\tau^3} f^{3/2}\left(\frac{sk}{k_0}\right) + \frac{k^{-3}}{\gamma_L^3 k_0^{9/2}}\frac{s^{9/2}}{t_0^3\tau^3}\frac{df^{3/2}}{dx}\frac{s}{k_0}\right]$$

$$= 2\gamma_L k^{\frac{13}{2}} \left[-\frac{3k^{-4}}{\gamma_L^3 k_0^{9/2} t_0^3} \frac{s^{9/2}}{\tau^3} f^{3/2} \left(\frac{sk}{k_0} \right) + \frac{k^{-3}}{\gamma_L^3 k_0^{11/2}} \frac{s^{11/2}}{t_0^3 \tau^3} \frac{df^{3/2}}{dx} \right]$$

$$= \frac{2\gamma_L k^{5/2} s^{9/2}}{\gamma_L^3 k_0^{9/2} t_0^3 \tau^3} \left[-3f^{3/2}(x) + \frac{3}{2} x f^{1/2}(x) \frac{df}{dx} \right], \; x = \frac{sk}{k_0}$$

$$= \frac{2}{\gamma_L^2} \left(\frac{ks}{k_0} \right)^{5/2} \frac{s^2}{k_0^2} \frac{1}{t_0^3 \tau^3} \left[-3f^{3/2}(x) + \frac{3}{2} x f^{1/2}(x) \frac{df}{dx} \right]$$

$$= \frac{2}{\gamma_L^2} \frac{x^{5/2} s^2}{k_0^2} \frac{1}{t^3} 3 f^{1/2}(x) \left[-f(x) + \frac{x}{2} \frac{df}{dx} \right] \qquad (4)$$

as $t_0 \tau = t$ and another term on the RHS of (1) e.g. is $2\nu \int_0^k k'^2 E(k', t)\, dk'$

$$= 2\nu \int_0^x \frac{x^2 k_0^2}{s^2 \gamma_L^2 k_0^3 t_0^2} \frac{s^3}{\tau^2} f(x) \frac{k_0}{s} dx \; \frac{sk'}{k_0} = x,$$

so that $dk' = \frac{k_0}{s} dx$

$$= 2\nu \int_0^x \frac{x^2}{\gamma_L^2 t_0^2 \tau^2} f(x) dx = \frac{2\nu}{\gamma_L^2 t_0^2 \tau^2} \int_0^x x^2 f(x) dx$$

$$= \frac{2\nu}{\gamma_L^2 t^2} \int_0^x x^2 f(x) dx \qquad (5)$$

Then from (1), using (3), (4) and (5), we get

$$\frac{s^2}{\gamma_L^2 k_0^3 t^3} \left[\int_0^x 2 \left(1 - \frac{\tau}{s} s_\tau \right) f(x') dx' - x s_\tau \frac{\tau}{s} f(x) \right]$$

$$= \frac{6s^2}{\gamma_L^2 k_0^2 t^3} x^{5/2} f^{1/2}(x) \left[-f(x) + \frac{x}{2} \frac{df}{dx} \right] + \frac{2\nu}{\gamma_L^2 t^2} \int_0^x x^2 f(x) dx$$

or

$$\int_0^x 2 \left(1 - \frac{\tau}{s} s_\tau \right) f(x') dx' - x s_\tau \frac{\tau}{s} f(x)$$

$$= 6x^{5/2} f^{1/2}(x) \left[-f(x) + \frac{x}{2} \frac{df}{dx} \right] + \frac{2\nu k_0^3 t}{s^2} \int_0^x x^2 f(x) dx \qquad (6)$$

Now, the case when Reynold's number is very high, we may assume that effect and ν is negligible and accordingly the last term of (6) may be dropped.

Thus, we have

$$\int_0^x 2 \left(1 - \frac{\tau}{s} s_\tau \right) f(s') dx' - x s_\tau \frac{\tau}{s} f(x)$$

$$= 6x^{5/2} f^{1/2}(x) \left[-f(x) + \frac{x}{2} \frac{df}{dx} \right] \qquad (7)$$

We now put in the above, $\tau \frac{s_\tau}{s} = c$ i.e. $\tau \frac{ds}{d\tau} = cs$ so that on integrating, we get $\log s = c \log \tau + \log a$, a being a constant i.e. $s = a\tau^c$. Then from (7), we have

$$\int_0^x 2(1-c) f(x') dx' - cx f(x)$$

$$= 6x^{5/2} f^{1/2}(x) \left[-f(x) + \frac{x}{2} \frac{df}{dx} \right] \qquad (8)$$

Asymptotic Behaviour : Asymptotic behaviour of the general solution $f(x)$ given by (8) as, (i) $x \to 0$ and (ii) $x \to \infty$.

Case I. $x \to 0$.

Let $f(x) = Ax^n + 0(x^n)$ where $0(x^n)$ is an infinitesimal of order n. Substituting this expression for $f(x)$ in (8), we have

$$\int_0^x 2(1-c)\left(Ax'^n + 0(x'^n) \right) dx' - cx \left(Ax^n + 0(x^n) \right)$$

$$= 6x^{5/2} \left(A^{1/2} x^{\frac{n}{2}} + 0\left(x^{\frac{n}{2}} \right) \right) \left(-Ax^n + 0(x^n) \right.$$

$$\left. + \frac{x}{2} Anx^{n-1} + 0\left(x^{n-1} \right) \right)$$

or

$$2A(1-c)\frac{x^{n+1}}{n+1} + 0(x^{n+1}) - Acx^{n+1} + 0(x^{n+1})$$

$$= 6A^{3/2} \left(x^{\frac{n+5}{2}} + 0\left(x^{\frac{n+5}{2}} \right) \right) \left(\left(\frac{n}{2} - 1 \right) x^n + 0(x^n) \right)$$

$$= 6A^{3/2} \left(\left(\frac{n}{2} - 1 \right) x^{\frac{3n+5}{2}} + 0\left(x^{\frac{3n+5}{2}} \right) \right)$$

Hence the co-efficient of x^{n+1} on the LHS above must vanish.

Therefore $\frac{2(1-c)}{n+1}-c=0 \Rightarrow 2-2c-nc-c=0$ i.e. $2-3c-cn=0 \Rightarrow n=\frac{2-3c}{c}$

Thus $f(x) \approx Ax^{\frac{2-3c}{c}}(x \to 0)$

Hence $E(k) \sim k^{\frac{2-3c}{c}}(k \to 0)$

Case II. $x \to \infty$

Here, we put $f(x) = Ax^t$, so that we get from (8)

$$\frac{x^{t+1}(3c-2+tc)}{t+1} = -6A^{1/2}x^{\frac{5+3t}{2}}\left(\frac{t}{2}-1\right) \to -6A^{1/2}\left(-\frac{11}{6}\right)$$

$$= 11A^{1/2} \text{as } t \to -5/3$$

Thus $f(x) \sim Ax^{-5/3}$ i.e. $E(K) \sim K^{-5/3}$ which agrees with Kolmogorof's 5/3rd law.

Concluding Remarks : We may write different forms for $f(x)$, as

$$f(x) \sim Ax(x \to 0) \text{ for } c = \frac{1}{2}$$

$$f(x) \sim Ax^2(x \to 0) \text{ for } c = \frac{2}{5}$$

$$f(x) \sim Ax^3(x \to 0) \text{ for } c = \frac{1}{3}$$

$$f(x) \sim Ax^4(x \to 0) \text{ for } c = \frac{2}{7}$$

As examined above, the asymptotic behaviour $f(x) \sim x^n(x \to 0)$, we have observed that four laws hold e.g.

$$f(x) \sim x(x \to 0),\ f(x) \sim x^2(x \to 0),$$

$$f(x) \sim x^3(x \to 0),\ f(x) \sim x^4(x \to 0)$$

for values of $c = \frac{1}{2}, \frac{2}{5}, \frac{1}{3}$ and $\frac{2}{7}$ respectively, On the other hand when $x \to \infty$, the asymptotic behaviour of $f(x)$ is given by $f(x) \sim x^{-5/3}(x \to \infty)$. Thus applying Sen's (1951) method, we obtain in class of asymptotic laws : $f(x) \sim x^n(x \to 0)$ is $f(x) \sim x,\ f(x) \sim x^2,\ f(x) \sim x^3,\ f(x) \sim x^4(x \to 0)$ all of which lead to the asymptotic behaviour of $f(x) \sim x^{-5/3}(x \to \infty)$, also in the present analysis.

In recent times, highly interesting works on the existence of multiple scaling laws and non-kolmogorov dissipation are reported. Layek and Sunita[9] obtained multiple scaling laws of decaying isotropic turbulence by adopting Lie symmetry sroup theory. Some of the laws support non-classical transfer of energy. The correction to the 5/3 law of energy spectrum function due to intermittency[12] has also been recently obtained by continuous compositions of symmetry group of transformations (see Layek and Sunita[10]) On the other hand, Vassilicos and co-workers experimentally observed non-Kolmogorov dissipation of energy in fractals grid generated turbulence (see Vassilicos[13]) Non-Kolmogorov dissipation has also been verified in free shear flows (see also Castro[14], Layek and Sunita[11]).

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1. L. F. Richardson, "Atmospheric diffusion shown on a distance-neighbour graph", *Proc. Roy. Soc.* **A110,** No. 756, 709-737.
2. A. N. Kolmogorov, "Local structure of Turbulence in an incompressible fluid at very high Reynolds numbers", *Dokl. Akad. Nauk. SSSR,* 30 No. 4, 299-303, (1941).
3. C. E. Leith. "Diffusion Approximation to Inertial Energy Transfer in Isotropic Turbulence." *The Phy. of Fluids,* Vol. 10, No. 7, 1409, 1967.
4. C. Connanghton and S. Nazarenko "Warm Cascade and Anomalous scaling in a Diffusion Model of Turbulence" *Phys. Rev. Lett.* **92**, (2004).
5. S. Thalabard, S. Nazarenko, S. Galtier and S. Medvedev S. "Anomalous Spectral Laws in Differential Models of Turbulence." *J. Phys. A. Math. Theor.* **48**, 1-21, (2015).
6. N. R. Sen "On Heisenberg's Spectrum of Turbulence." *Bull. Cal. Math. Soc.,* **43**, No. 1, 1-7, (1951).
7. N. R. Sen "The Modern Theory of Turbulence," Professorship Lecture for 1951, *Indian Association for the Cultivation of Science,* 1-39, April, (1956).

8. N. R. Sen "Isotropic Turbulence Preserving Similarity." *J. Indian Math. Soc.* **Vol. 24**, Nos. 3 & 4, (1960).

9. G. C. Layek and Sunita : "On the nature of multitude scalings in decaying isotropic turbulence", *Int. J. of Non-linear Mechanics,* 95 (2017).

10. G. C. Layek and Sunita : Multi-fractal cascade symmetry model for fully developed turbulence, Fractals vol-26 No. 4 (2018) 1850070.

11. G. C. Layek and Sunita : Multitude scaling laws in axisymmetric turbulent wake, *Phys. Fluids,* 30, 035101(2018).

12. U. Frisch, Turbulence ; The Legacy of A. N. Kolmogorov, Cambridge University press, Cambridge U. K. (1995).

13. J. C. Vassilicos, "Dissipation in Turbulent Flows", *Annu Rev. Fluid Much*, **47**, 95-114, (2015).

14. I. P. Castro "Dissipative distinctions" *J. Fluid Much.* **788**, 1-4, (2016).